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Triebel-Lizorkin Spaces Estimates for Evolution Equations with Structure Dissipation

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ABSTRACT

Triebel-Lizorkin space estimates for evolution equations with structure dissipation

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This work is concerned with the long time decay estimates of the generalized heat equations and the generalized wave equations in the homogeneous Triebel-Lizorkin spaces. We first extend the known results for the generalized heat equations in the real Hardy spaces. We also extend the known results for the generalized wave equations with structure dissipation in the real Hardy spaces.

The main tools employed are the decomposition of the unit, duality property in Triebel-Lizorkin spaces and the multiplier theorems in different function spaces such as Lebesgue spaces, real Hardy spaces and Triebel-Lizorkin spaces.

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Chapter 1

Introduction

Partial differential equations (PDEs) are applied widely, since a rich variety of physical, chemical, biological, probabilistic and statistical phenomena can be modeled by PDEs. One of the core problems in PDEs is to study the well-posedness which has attracted much attention in the history of PDEs. We say that a given problem for a partial differential equation is well-posed if
(i) the problem in fact has a solution;
(ii) this solution is unique;
(iii) the solution depends continuously on the data given in the problem.
Namely, well-posedness is to study of the existence, uniqueness and stability of the solutions to PDEs. To get the uniqueness, very often we have to estimate the solutions in some sense. In this thesis, we are going to study the behavior of the solutions to the generalized heat equations and the generalized wave equations. To be more specific, we study the long time decay estimates of the solutions to the generalized heat equations and the generalized wave equations which are studied in Chapter 3 and Chapter 4 in details.

There are many tools to study PDEs. Here we mainly depend on the multiplier theorem (Fourier multiplier), interpolation theorem and duality theorem. Since the milestone works of S.G. Michilin and L.Hömander, Fourier multipliers on function spaces have attracted much attention for their own sake. For instance, Fourier multipliers on Lebesgue spaces [12], real Hardy space [25], homogeneous Besov spaces [5] and homogeneous Triebel-Lizorkin spaces [4] have been treated extensively. For further details, we refer the reader to Chapter 2 in this thesis. Interpolation methods exist essentially two methods: the real interpolation method by J.L. Lions and J. Peetre, and the complex interpolation method by J.-L. Lions, A.P. Calderón and S.G. Krejn. An extensive treatment of these abstract methods and many references have been given in [31], also J. Bergh, J. Löfström [2]. Originally, both methods were developed in the framework of Banach space theory. However, it is not difficult to see (and well-known nowadays) that the real method (at least some of its crucial assertions) can be extended immediately to quasi-Banach spaces. Dual space is an important concept in functional analysis since dual spaces are used to describe measures, distributions and other more complicated function spaces. For instance, the dual space of $H^1(\mathbb{R}^n)$ is BMO space [8].

As we all know that the properties of the solutions to the same PDE in different function spaces can be totally different. In order to study PDEs, it is important to find an appropriate function space. In other words, function spaces play a crucial role in PDEs. Since the 1930's, more sophisticated function spaces have been used in the theory of partial differential equations, in the first place the Hölder spaces and the Sobolev spaces. Later on, especially in the 1950's and 1960's, many new spaces were created and investigated, e.g. Besov spaces, Lebesgue spaces, Hardy spaces and the space BMO which is the dual space of $H^1(\mathbb{R}^n)$. Additionally, function spaces play an important part in both classical and modern mathematics. Spaces whose elements are continuous, or differentiable, or p-integrable functions are of interest for their own sake. For example, the embedding theory, duality theory and interpolation theory were studied extensively in different function spaces. Recently, there are many literatures involving the study of decompositions or characterizations of the function spaces, for instance, dyadic decomposition, atomic decomposition, wavelet decomposition and Riesz transform characterization. In this thesis, we are mainly focus on the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ and the homogeneous Triebel-Lizorkin $\dot{F}^s_{p,r}(\mathbb{R}^n)$ space with the classical dyadic decomposition since these two families are interesting in their own right, but their importance also stems from the fact that several of the classical function spaces such as Lebesgue, Hardy, BMO, Sobolev, and Höder spaces can be recovered as special cases.

The structure of this thesis is as follows.
Chapter 2 contains some preliminary knowledge. This summary contains no new results, but most of the facts which are required for later chapters.
Chapter 3 presents the results for the generalized heat equations in Triebel-Lizorkin spaces and Besov spaces. In particular, we derived the long time decay estimates for the solution of the generalized heat equations in Triebel-Lizorkin spaces. Thus, we extend the known results for the generalized heat equations in real Hardy spaces.
Chapter 4 is the central part of this thesis which studies the generalized wave equations with

structure dissipation in Triebel-Lizorkin spaces. More precisely, we obtained the long time decay estimates for the solution of the generalized wave equations with structure dissipation in Triebel-Lizorkin spaces. So we extend the known results for the generalized wave equations with structure dissipation in real Hardy spaces.

Chapter 5 talks about our future research which is related to this thesis.

Chapter 6 contains two appendices. One of them is the well-known Riesz characterization of the real Hardy spaces. Another one is the proof of the duality property in the homogeneous Triebel-Lizorkin spaces. They are both quite useful to get the long time decay estimate for the generalized wave equations with structure dissipation in Triebel-Lizorkin spaces, especially for the non-effective case when $\delta < \sigma < 2\delta$.

Chapter 2

Preliminaries

In this chapter, we review some function spaces and their corresponding multiplier theorems.

2.1 Function spaces

2.1.1 Lebesgue spaces

Let dx be the Lebesgue measure on $\mathbb{R}^n$. The Lebesgue space $L^p(\mathbb{R}^n), 0 < p < \infty$, is the set of all measurable functions $f(x)$ satisfying

$$\|f\|_{L^p(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty. \tag{2.1}$$

As well known, when $1 \le p < \infty$, The Lebesgue space $L^p(\mathbb{R}^n)$ is a Banach space. And its dual space is $L^q(\mathbb{R}^n)$, where q satisfies the condition $\frac{1}{p} + \frac{1}{q} = 1$. However, when $0 < p < 1$, its dual space is trivial, namely, it contains only one constant function 0.

For $0 < p < \infty$, the space weak $L^p(X, \mu)$ is defined as the set of all measurable functions $f(x)$ such that

$$\|f\|_{L^{p,\infty}} = \inf \Big\{ C > 0 : \mu(\{x \in X : |f(x)| > \zeta\}) \le \frac{C^p}{\zeta^p} \text{ for all } \zeta > 0 \Big\} < \infty. \tag{2.2}$$

2.1.2 Real Hardy spaces $H^p(\mathbb{R}^n)$

We recall how the real Hardy spaces $H^p(\mathbb{R}^n)$ are presented by Fefferman and Stein in [8]. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ which is the Schwartz function space, with integral equal to 1. For $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the maximal function $M_\varphi u$ by

$$M_\varphi u(x) = \sup_{0<t<\infty} |(u * \varphi_t)(x)|,$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$, and $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$.

Definition 1 *Let $0 < p < \infty$. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^p(\mathbb{R}^n)$ if and only if $M_\varphi u \in L^p(\mathbb{R}^n)$, i.e.,*

$$\|u\|_{H^p} := \|M_\varphi u\|_{L^p} < \infty.$$

For $p = \infty$, we set $H^\infty(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$, for the sake of simplicity.

The definition of $H^p(\mathbb{R}^n)$ is independent of the choice of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. For $p = 1$, $\|u\|_{H^1}$ is a norm and $H^1(\mathbb{R}^n)$ is a normed space densely contained in $L^1(\mathbb{R}^n)$. For $p > 1$, $\|u\|_{H^p}$ is a norm which is equivalent to the usual L^p norm $\|u\|_{L^p}$ and we denote $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, by abusing notation. For $0 < p \leq 1$, the space $H^p(\mathbb{R}^n)$ is a complete metric space with the distance

$$d(u, v) = \|u - v\|^p_{H^p}, \qquad u, v \in H^p(\mathbb{R}^n).$$

Although $H^p(\mathbb{R}^n)$ is not locally convex for $0 < p < 1$ and $\|u\|_{H^p}$ is not truly a norm (it is a quasi-norm [30]), we will still refer to $\|u\|_{H^p}$ as the "norm" of u, as it is customary.

The dual space of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation. Here we briefly recall that $BMO(\mathbb{R}^n)$ is the space of all locally integrable functions f satisfying

$$\|f\|_{BMO} := \sup \frac{1}{|B|}\int_B \left| f(x) - \frac{1}{|B|}\int_B f(t)dt \right| dx < \infty,$$

where the sup is taken over all balls B with center x in $\mathbb{R}^n$. It is known that the space $L^\infty(\mathbb{R}^n)$ is a proper subspace of $BMO(\mathbb{R}^n)$ and

$$log|x| \in BMO(\mathbb{R}^n) \backslash L^\infty(\mathbb{R}^n).$$

2.1.3 Triebel-Lizorkin spaces and Besov spaces

For a systematic approach, we also recall the definitions of Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$. Let $\mathcal{A}$ denote the class of Schwartz functions φ on $\mathbb{R}^n$ such that their Fourier transforms $\hat{\varphi}$ have support in $\{1/2 \leq |\xi| \leq 2\}$ and $|\hat{\varphi}(\xi)| \geq c > 0$ for $3/5 \leq |\xi| \leq 5/3$. Given a triple of parameters $(s, p, q) \in \mathbb{R} \times (0, \infty) \times (0, \infty]$, we recall that

([26], [30]) a tempered distribution f belongs to the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$, modulo polynomials, if the quasi-norm

$$\|f\|_{\dot{F}^s_{p,q}} = \Big\| \Big(\sum_{j\in\mathbb{Z}} (2^{js}|f*\varphi_j|^q \Big)^{1/q} \Big\|_{L^p} \tag{2.3}$$

is finite, with the usual interpretation for $q=\infty$. An extension to the case $p=\infty$ reads as

$$\|f\|_{\dot{F}^s_{\infty,q}} = \sup_Q \Big(\frac{1}{|Q|} \int_Q \sum_{2^{-j}\le l(Q)} (2^{js}|f*\varphi_j(x)|^q dx \Big)^{1/q}, \tag{2.4}$$

where the supremum is taken over all dyadic cubes Q ([9]).

A tempered distribution f belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ ([10], [26], [32]), modulo polynomials, if the quasi-norm

$$\|f\|_{\dot{B}^s_{p,q}} = \Big(\sum_{j\in\mathbb{Z}} (2^{js}\|f*\varphi_j\|_{L^p})^q \Big)^{1/q} \tag{2.5}$$

is finite, where $\varphi_j(x) = 2^{jn}\varphi(2^j x)$. A different choice of φ in all definitions above yields equivalent quasi-norms as long as it is taken from the class $\mathcal{A}$.

2.2 Multipliers

Fourier multipliers form one of the fundamental and most important classes of operators in harmonic analysis. Their importance is emphasized by their close link to partial differential operators through the Fourier transform, and there has been a continuous interest in the study of boundedness properties of multipliers on L^p and other spaces since the work by Marcinkiewicz [23], Mihlin [24] and Hörmander [13]. Roughly speaking, Fourier multiplier operator is a type of linear operator which is a special case of a pseudo-differential operator. For more information, we refer the readers to [29]. Occasionally, the term multiplier operator itself is shortened simply to be multiplier. Multiplier operators can be defined on any group G for which the Fourier transform is also defined (in particular, on any locally compact Abelian group). Here we take the following definition:

$$T_m(f)(x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi = (m\hat{f})^\vee, \tag{2.6}$$

where T_m is called multiplier operator and $m(\xi)$ is said to be the multiplier/symbol of T_m. As usual, if $f\in\mathcal{S}$, the definitions of Fourier transform of f and its inverse are given by

$$(\mathcal{F}(f))(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}^n, \tag{2.7}$$

and

$$(\mathcal{F}^{-}(f))(x) = \check{f}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) e^{ix\xi} d\xi, \quad x \in \mathbb{R}^n. \tag{2.8}$$

To illustrate the importance of the multiplier, we look at the following several classical examples.

Example 2.1. Derivative d/dx

$$\left(\frac{df(x)}{dx}\right)^{\wedge} = 2\pi i\xi \hat{f}(\xi).$$

From this fact, we see that a differential operator converts to a polynomial in the frequency space after taking Fourier transform.

Example 2.2. The Hilbert transform H is defined as

$$Hf := p.v.\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

Its Fourier multiplier is $-isgn(\xi)$.

Example 2.3. The Riesz transforms $R_j, j = 1, 2, \cdots, n$

$$R_j f := p.v.c_n \int_{\mathbb{R}^n} \frac{f(y)(x_j - y_j)}{|x-y|^n} dy.$$

Its multiplier is $-i\frac{\xi_j}{|\xi|}$, for each $j = 1, 2, \cdots, n$, and $p.v.$ means the Cauchy principle value, c_n is a constant which depends on the dimension of $\mathbb{R}^n$.

From **Example 2.2** and **Example 2.3**, we can see that Riesz transforms R_j are extensions of the Hilbert transform H in higher dimensions. And they are both classical examples of singular integrable operator which is a branch of harmonic analysis. These examples begin to show the importance of multipliers in analysis. In this thesis, we will employ the generalized multipliers in Triebel-Lizorkin spaces [4] and Besov spaces [5].

In what follows, we recall the multipliers on different function spaces. The first ingredient is the celebrated Mikhlin- Hörmander multiplier theorem for Lebesgue spaces $L^p(\mathbb{R}^n)$.

2.2.1 The multiplier on $L^p(\mathbb{R}^n)$

Let $\mathcal{M}_p(\mathbb{R}^n), 1 \leq p < \infty$ denote the space of all bounded functions m on $\mathbb{R}^n$ such that the Fourier multiplier operator

$$T_m(f) = (m\hat{f})^\vee, \qquad f \in \mathcal{S},$$

is bounded on $L^p(\mathbb{R}^n)$ (or is initially defined in a dense subspace of $L^p(\mathbb{R}^n)$ and has a bounded extension on the whole space). The norm of m in $\mathcal{M}_p(\mathbb{R}^n)$ is defined by

$$\|m\|_{\mathcal{M}_p} = \|T_m\|_{L^p \to L^p}.$$

The norm space $\mathcal{M}_p(\mathbb{R}^n)$ are nested, that is, for $1 \leq p \leq q \leq 2$ we have

$$\mathcal{M}_1 \subseteq \mathcal{M}_p \subseteq \mathcal{M}_q \subseteq \mathcal{M}_2 = L^\infty.$$

Now let us recall the celebrated Mikhlin- Hörmander multiplier theorem for Lebesgue spaces $L^p(\mathbb{R}^n)$[12].

Theorem 1 *([12]) Let $m(\xi)$ be a complex-valued bounded function on $\mathbb{R}^n \setminus \{0\}$ that satisfies for some $A < \infty$*

$$\Big(\int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 d\xi \Big)^{\frac{1}{2}} \leq AR^{\frac{n}{2}-|\alpha|} < \infty, \tag{2.9}$$

for all multi-indices $|\alpha| \leq [n/2] + 1$ and all $R > 0$.

Then for all $1 < p < \infty, m$ lies in $\mathcal{M}_p(\mathbb{R}^n)$ and the following estimate is valid:

$$\|m\|_{\mathcal{M}_p} \leq C_n \max(p, (p-1)^{-1})(A + \|m\|_{L^\infty}). \tag{2.10}$$

Moreover, the operator $f \mapsto (\hat{f}m)^\vee$ maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with norm at most a dimensional constant multiple of $A + \|m\|_{L^\infty}$.

We remark that in most applications, the condition on m in above theorem appears in the form

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{-\alpha}, \tag{2.11}$$

which should be, in principle, easier to verify.

The following proposition summarizes some simple properties of multipliers on Lebesgue spaces $L^p(\mathbb{R}^n)$.

Proposition 1 *([12]) For all $m \in \mathcal{M}_p, 1 \le p < \infty, x \in \mathbb{R}^n$, and $h > 0$ we have*

$$\|\tau^x(m)\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p},$$

$$\|\delta^h(m)\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p},$$

$$\|\tilde{m}\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p},$$

$$\|e^{2\pi i(\cdot)\cdot x}(m)\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p},$$

$$\|m \circ A\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p},$$

where $(\tau^x m)(\cdot) = m(\cdot - x)$, $(\delta^h m)(\cdot) = m(h\cdot)$ and A is an orthogonal matrix $\mathbb{R}^n \to \mathbb{R}^n$.

2.2.2 The multiplier on $H^p(\mathbb{R}^n)$

We list briefly some of the multiplier theorems to be used in the sequel. The first ingredient is a version of the celebrated Mikhlin-Hörmander multiplier theorem on Hardy spaces.

Theorem 2 *([25]) Let $0 < p < \infty$ and $k = max\{[n(1/p - 1/2)] + 1, [n/2] + 1\}$. Suppose that $m \in \mathcal{C}^k(\mathbb{R}^n \setminus \{0\})$ and*

$$|\partial_\xi^\beta m(\xi)| \le C|\xi|^{-\beta}, \quad |\beta| \le k.$$

Then T_m is continuously bounded from $H^p(\mathbb{R}^n)$ into itself.

Variants of this theorem assuming conditions on the support of $m(\xi)$ can be found in [25]. In particular, we recall the following two main theorems in [25]. In those two theorems, $\mathcal{M}(H^p(\mathbb{R}^n))$ denotes the set of all Fourier multipliers on $H^p(\mathbb{R}^n)$.

Theorem 3 *([25]) Let $a \ge 0, b \ge 0, 0 < p_0 < 2, na(1/p_0 - 1/2) = b$, and $k = \max\{[n(1/p_0 - 1/2)] + 1, [n/2] + 1\}$. Suppose that $m \in C^k(\mathbb{R}^n)$, $m(\xi) = 0$ in a neighborhood of the origin, and*

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha m(\xi)\right| \le |\xi|^{-b}(A|\xi|^{a-1})^{|\alpha|}, \quad |\alpha| \le k, \tag{2.12}$$

with some constant $A \ge 1$. Then $m \in \mathcal{M}(H^p(\mathbb{R}^n))$ and

$$\|m\|_{\mathcal{M}(H^p(\mathbb{R}^n))} \le CA^{n(1/p-1/2)} \tag{2.13}$$

for $2 \ge p \ge p_0$, where C is a constant independent of A.

Theorem 4 *([25]) Let* $c \geq 0, d \geq 0, 0 < p_0 < 2, nd(1/p_0 - 1/2) = c$, *and* $k = \max\{[n(1/p_0 - 1/2)] + 1, [n/2] + 1\}$. *Suppose that* $m \in C^k(\mathbb{R}^n \setminus \{0\}), m(\xi) = 0$ *if* $|\xi| \geq 1$, *and*

$$\Big|(\frac{\partial}{\partial \xi})^{\alpha} m(\xi)\Big| \leq |\xi|^c (A|\xi|^{-1-d})^{|\alpha|}, \quad |\alpha| \leq k, \tag{2.14}$$

with some constant $A \geq 1$. *Then* $m \in \mathcal{M}(H^p(\mathbb{R}^n))$ *and*

$$\|m\|_{\mathcal{M}(H^p(\mathbb{R}^n))} \leq CA^{n(1/p-1/2)}. \tag{2.15}$$

for $2 \geq p \geq p_0$, *where* C *is a constant independent of* A.

Another useful property of Hardy spaces is a pointwise estimate (see Corollary 7.21 in [11] or (77) [6]) for the Fourier transform of H^p functions, with $p \in (0, 1]$:

$$|\hat{f}(\xi)| \leq C|\xi|^{n(\frac{1}{p}-1)} \|f\|_{H^p}, \quad p \in (0, 1]. \tag{2.16}$$

Moreover, the following integral estimate (see Corollary 7.23 in [11] or (78) [6]) holds:

$$\Big(\int_{\mathbb{R}^n} |\xi|^{n(p-2)} |\hat{f}(\xi)|^p d\xi\Big)^{\frac{1}{p}} \leq C\|f\|_{H^p}, \quad p \in (0, 1]. \tag{2.17}$$

2.2.3 The multiplier on $\dot{F}^s_{p,r}(\mathbb{R}^n)$ and $\dot{B}^s_{p,r}(\mathbb{R}^n)$

Finally, let us recall what kinds of functions m can be the multiplier on $\dot{F}^{\alpha}_{p,r}$ [4] and $\dot{B}^{\alpha}_{p,r}$ [5]. Give a positive integer l and $\alpha \in \mathbb{R}, m \in C^l(\mathbb{R}^n \setminus \{0\})$ and

$$\sup_{R>0} \Big[R^{-n+2\alpha+2|\sigma|} \int_{R<|\xi|<2R} |\partial^{\sigma}_{\xi} m(\xi)|^2 d\xi\Big] \leq A_{\sigma}, |\sigma| \leq l. \tag{2.18}$$

When $\alpha = 0$, it is known as the Hörmander condition. Typical examples are given by the symbols of the Riesz singular integrals R_j (see Appendix). When $\alpha \neq 0$, one of the typical examples is $m(\xi) = |\xi|^{-\alpha}$ which satisfies condition (2.18) for every positive integer l. For further multiplier theorems in $\dot{F}^s_{p,r}(\mathbb{R}^n)$ and $\dot{B}^s_{p,r}(\mathbb{R}^n)$ spaces, we refer the readers to ([4], [5]) and next chapter in this thesis. The following is one of the multiplier theorems in Triebel-Lizorkin spaces.

Theorem 5 *([4] Theorem 5.2) Given* $\alpha \in \mathbb{R}, 0 < p < \infty$ *and* $0 < r \leq \infty$*, let* β *be any real number with* $\beta < \alpha$ *and let* p_* *be determined by*

$$\beta - n/p_* = \alpha - n/p, \qquad 0 < p_* \leq \infty. \tag{2.19}$$

Assume that m *satisfies condition (2.18) with*

$$l > \begin{cases} \max(n/p, n/r) + n(1/2 - 1/r), & if\ 2 \leq r \leq \infty, \\ \max(n/p, n/2), & if\ 0 \leq r \leq 2. \end{cases} \tag{2.20}$$

Then T_m *maps* $\dot{F}^0_{p,r}$ *boundedly into* $\dot{F}^\beta_{p_*,q}$ *for any* $0 < q \leq \infty$ *with*

$$\|T_m f\|_{\dot{F}^\beta_{p_*,q}} \preceq \|f\|_{\dot{F}^0_{p,r}}. \tag{2.21}$$

Chapter 3

Heat equations

3.1 Introduction

We begin with studying the initial value problem for the generalized heat equation

$$\begin{cases} u_t + (-\Delta)^\sigma u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1}$$

where $\sigma \in (0,\infty)$ and $(-\Delta)^\sigma f := \mathcal{F}^{-1}(|\xi|^{2\sigma}\mathcal{F}f)$, $\mathcal{F}(\wedge)$ being the Fourier transform with respect to the spatial variable x and $\mathcal{F}^{-1}(\vee)$ being the inverse of the Fourier transform. The initial data u_0 belongs to some Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ $(\dot{F}^s_{p,q})$ or some Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ $(\dot{B}^s_{p,q})$. Equation (3.1) is significantly interesting in both physics and partial differential equations (PDEs), since it is the classical heat equation when $\sigma = 1$ and the Poisson equation when $\sigma = 1/2$. Recently, for the non-homogeneous heat equations, in [3] the authors obtained the estimates of the solution of the heat equation in the real Hardy space $H^p(\mathbb{R}^n)$ and studied the well-posedness of the Cauchy problem related to the heat equation.

As we know, an efficient way to solve PDEs is trying to transform the PDEs to the Ordinary Differential Equations (ODEs). By taking the Fourier transform with respect to the spatial variable x, and then solving the corresponding (ODEs) with respect to time t, we get the solution of the Cauchy problem (3.1) which is formally written by

$$u(t,x) = e^{-t(-\Delta)^\sigma} u_0(x), \tag{3.2}$$

where for each fixed t, $e^{-t(-\Delta)^\sigma}$ is a Fourier multiplier with symbol $m(t,\xi) = e^{-t|\xi|^{2\sigma}}$. Thus, we may rewrite the solution of the Cauchy problem (3.1) as Fourier multiplier operator with $m(t,\xi) = e^{-t|\xi|^{2\sigma}}$:

$$u(t,x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\sigma}}\mathcal{F}u_0(\xi))(t,x) = T_m u_0(x). \tag{3.3}$$

In this chapter, our aim is to estimate the solution to (3.1). The basic tool employed is the Fourier multipliers whose symbols satisfy a generalization of Hörmander's condition (2.18) on m ([4], [5]).

From then on, the notation $A \preceq B$ means that there is a positive constant C such that $A \leq CB$. $A \approx B$ means that there exist positive constants c and C such that $cA \leq B \leq CA$. The constant C and c may depend on the parameters but not on the variable quantities involved and may vary from line to line. The set of all Fourier multipliers for $\dot{F}^{\alpha}_{p,r}$ and $\dot{B}^{\alpha}_{p,r}$ are denoted by $\dot{\mathfrak{m}}_{p,r}$ and $\dot{\mathfrak{m}}_{p}$ respectively.

3.2 Some results in $\dot{F}^s_{p,r}(\mathbb{R}^n)$ and $\dot{B}^s_{p,r}(\mathbb{R}^n)$

To be convenient for the readers, in this section, we shall collect some known results in $\dot{F}^s_{p,r}(\mathbb{R}^n)$ and $\dot{B}^s_{p,r}(\mathbb{R}^n)$ and also prove some lemmas which will be used later on.

Proposition 2 *([4] Proposition 2.1(1)) Let $0 < p, r \leq \infty$. For $s \in \mathbb{R}, \dot{F}^s_{p,r} = I_\alpha(\dot{F}^0_{p,r})$ so that $f \in \dot{F}^s_{p,r}$ if and only if there exists a unique $g \in \dot{F}^0_{p,r}$ such that*

$$f = I_s(g) \qquad \textit{and} \qquad \|f\|_{\dot{F}^s_{p,r}} \approx \|g\|_{\dot{F}^0_{p,r}}.$$

There is a counterpart of $\dot{B}^s_{p,r}(\mathbb{R}^n)$, which means, with the same assumption above, the following holds:

For $s \in \mathbb{R}$, $\dot{B}^s_{p,r} = I_s(\dot{B}^0_{p,r})$ so that $f \in \dot{B}^s_{p,r}$ if and only if there exists a unique $g \in \dot{B}^0_{p,r}$ such that

$$f = I_s(g) \qquad \text{and} \qquad \|f\|_{\dot{B}^s_{p,r}} \approx \|g\|_{\dot{B}^0_{p,r}},$$

where I_s denotes the Riesz potential defined by

$$(I_s f)\hat{}(\xi) = |\xi|^{-s}\hat{f}(\xi).$$

Owning to Proposition 2.1 [4], to prove that a convolution operator T is bounded on the spaces $\dot{F}^s_{p,q}$ and $\dot{B}^s_{p,q}$, it suffices to show its boundedness on $\dot{F}^0_{p,q}$ and $\dot{B}^0_{p,q}$. For example, on the space $\dot{F}^s_{p,q}$, once we prove

$$\|Tf\|_{\dot{F}^0_{p,r}} \preceq \|f\|_{\dot{F}^0_{p,r}} \tag{3.4}$$

we have

$$\begin{aligned}\|Tf\|_{\dot{F}^s_{p,r}} &= \|TI_s(g)\|_{\dot{F}^s_{p,r}} \\ &= \|I_sTg\|_{\dot{F}^s_{p,r}} \\ &\approx \|Tg\|_{\dot{F}^0_{p,r}} \preceq \|g\|_{\dot{F}^0_{p,r}} \\ &\approx \|f\|_{\dot{F}^s_{p,r}}.\end{aligned}$$

Next is one of the known results of Fourier multipliers on Triebel-Lizorkin space $\dot{F}^s_{p,r}$ which is also called the lifting property.

Theorem 6 *([4] Theorem 5.1) (lifting property). Let* $\alpha,\ \gamma \in \mathbb{R},\ 0 < p < \infty$ *and* $0 < r \leq \infty$. *Suppose that* m *satisfies condition (2.18) with* $l > \max(n/p, n/r) + n/2$. *Then*

$$\|T_m f\|_{\dot{F}^{\alpha+\gamma}_{p,r}} \preceq \|f\|_{\dot{F}^{\gamma}_{p,r}}. \tag{3.5}$$

If $l > \lambda + n/2$ *and* λ *is sufficiently large, then*

$$\|T_m f\|_{\dot{F}^{\alpha+\gamma}_{\infty,r}} \preceq \|f\|_{\dot{F}^{\gamma}_{\infty,r}}. \tag{3.6}$$

Corollary 1 *([4] Corollary 5.1) Given* $\alpha,\ \gamma \in \mathbb{R},\ 0 < p < \infty$ *and* $0 < r \leq \infty$, *let* β *be any real with* $\beta < \alpha + \gamma$ *and let* p_* *be determined by*

$$\beta - n/p_* = \alpha + \gamma - n/p, \qquad 0 < p_* \leq \infty. \tag{3.7}$$

Assume that m *satisfies condition (2.18) with (2.20). Then* T_m *maps* $\dot{F}^{\gamma}_{p,r}$ *boundedly into* $\dot{F}^{\beta}_{p_*,q}$ *for any* $0 < q \leq \infty$ *with*

$$\|T_m f\|_{\dot{F}^{\beta}_{p_*,q}} \preceq \|f\|_{\dot{F}^{\gamma}_{p,r}}. \tag{3.8}$$

Also, we have some results of Fourier multipliers on the Besov space $\dot{B}^s_{p,r}$.

Theorem 7 *([5] Theorem 1.1) Given* $\alpha \in \mathbb{R}, 0 < p < \infty$ *and* $0 < r \leq \infty$, *let* β *be any real number with* $\beta < \alpha$ *and let* p_* *be determined by*

$$\beta - n/p_* = \alpha - n/p, \qquad 0 < p_* \leq \infty. \tag{3.9}$$

Assume that m *satisfies condition (2.18) with* $l > n(1/p+1/2)$, *then* T_m *maps* $\dot{B}^0_{p,r}$ *boundedly into* $\dot{B}^{\beta}_{p_*,q}$ *for any* $0 < q \leq \infty$ *with*

$$\|T_m f\|_{\dot{B}^{\beta}_{p_*,q}} \preceq \|f\|_{\dot{B}^0_{p,r}}. \tag{3.10}$$

Corollary 2 *Given* $\alpha, \gamma \in \mathbb{R}, 0 < p < \infty$ *and* $0 < r \leq \infty$, *let* β *be any real with* $\beta < \alpha + \gamma$ *and let* p_* *be determined by*

$$\beta - n/p_* = \alpha + \gamma - n/p, \qquad 0 < p_* \leq \infty. \tag{3.11}$$

Assume that m *satisfies condition (2.18) with* $l > n(1/p + 1/2)$, *then* T_m *maps* $\dot{B}^{\gamma}_{p,r}$ *boundedly into* $\dot{B}^{\beta}_{p_*,q}$ *for any* $0 < q \leq \infty$ *with*

$$\|T_m f\|_{\dot{B}^{\beta}_{p_*,q}} \preceq \|f\|_{\dot{B}^{\gamma}_{p,r}}. \tag{3.12}$$

To prove Corollary 2, we need the following lemma.

Lemma 1 *If* m *satisfies condition (2.18), then* $\tilde{m} = m(\xi)|\xi|^{-\gamma}$ *satisfies condition (2.18) by replacing* α *by* $\tilde{\alpha} = \alpha + \gamma$.

Proof. For any $0 \leq |\sigma| \leq l$, using Leibniz's formula, we have

$$\partial^{\sigma}_{\xi} \tilde{m} = \partial^{\sigma}_{\xi}(m(\xi)|\xi|^{-\gamma}) = \sum_{\beta < \sigma} C_{\beta} \partial^{\beta}_{\xi} m \; \partial^{\sigma-\beta}_{\xi} |\xi|^{-\gamma}. \tag{3.13}$$

Thus,

$$\begin{aligned}
R^{-n+2(\alpha+\gamma)+2|\sigma|} \int_{R<|\xi|<2R} |\partial^{\sigma}_{\xi} \tilde{m}|^2 d\xi &\preceq R^{-n+2(\alpha+\gamma)+2|\sigma|} \int_{R<|\xi|<2R} |\sum_{\beta<\sigma} C_{\beta} \partial^{\beta}_{\xi} m \; \partial^{\sigma-\beta}_{\xi} |\xi|^{-\gamma}|^2 d\xi \\
&\preceq R^{-n+2(\alpha+\gamma)+2|\sigma|} \sum_{\beta<\sigma} C_{\beta} \int_{R<|\xi|<2R} |\partial^{\beta}_{\xi} m|^2 |\partial^{\sigma-\beta}_{\xi} |\xi|^{-\gamma}|^2 d\xi \\
&\preceq R^{-n+2(\alpha+\gamma)+2|\sigma|} \sum_{\beta<\sigma} C_{\beta} \int_{R<|\xi|<2R} |\partial^{\beta}_{\xi} m|^2 \frac{1}{|\xi|^{2(\gamma+|\sigma|-|\beta|}} d\xi \\
&\preceq R^{-n+2(\alpha+\gamma)+2|\sigma|} R^{-2\gamma-2|\sigma|+2|\beta|} \sum_{\beta<\sigma} C_{\beta} \int_{R<|\xi|<2R} |\partial^{\beta}_{\xi} m|^2 d\xi \\
&= R^{-n+2\alpha+2|\beta|} \sum_{\beta<\sigma} C_{\beta} \int_{R<|\xi|<2R} |\partial^{\beta}_{\xi} m|^2 d\xi \\
&\preceq A_{\sigma},
\end{aligned}$$

where we used the assumption imposed on m in the last inequality. So we end the proof of Lemma 1.

Next we prove Corollary 2.

Proof. For any $f \in \dot{B}^{\gamma}_{p,r}$, by Proposition 2.1 $\dot{B}^{\gamma}_{p,r} = I_{\gamma}(\dot{B}^{0}_{p,r})$, there exists a unique $g \in \dot{B}^{0}_{p,r}$

such that $f = I_\gamma g$ with $\|f\|_{\dot{B}^{\gamma}_{p,r}} \approx \|g\|_{\dot{B}^{0}_{p,r}}$. Then, we have

$$\begin{aligned} \|T_m f\|_{\dot{B}^{\beta}_{p_*,r}} &= \|T_m I_\gamma(g)\|_{\dot{B}^{\beta}_{p_*,r}} \\ &= \|T_{\tilde{m}} g\|_{\dot{B}^{\beta}_{p_*,r}} \\ &\preceq \|g\|_{\dot{B}^{0}_{p,r}} \\ &\approx \|f\|_{\dot{B}^{\gamma}_{p,r}}, \end{aligned}$$

where we employed Lemma 1. So Corollary 2 is proved.

Before we state our main results in this chapter, we prove another lemma related to the relationship between the multipliers $\dot{\mathfrak{m}}_{p,r}$ and $\dot{\mathfrak{m}}_p$.

Lemma 2 *Let $0 < p < \infty, 0 < r \leq \infty$, then $\dot{\mathfrak{m}}_{p,r} \subset \dot{\mathfrak{m}}_p$.*

Our proof is based on the results of Theorem 5.1 in [17] which is stated as follows. Before we give the theorem, let us introduce some notations. Let $(\cdot,\cdot)_{\theta,r}$ stand for the standard real interpolation bracket. More specifically, consider a compatible couple of quasi-Banach spaces $X_0,\ X_1$. Given $a \in X_0 + X_1$ and $0 < t < \infty$, Peetre's K-functional is defined by

$$K(t, a; X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 \in X_0, x_1 \in X_1 \text{ such that } \ a = x_0 + x_1\}. \quad (3.14)$$

Then we introduce the real interpolation spaces as

$$(X_0, X_2)_{\theta,r} := \Big\{a \in X_0 + X_1 : \|a\|_{(X_0,X_2)_{\theta,r}} := \Big(\int_0^\infty (t^{-\theta} K(t, a; X_0, X_1))^r \frac{dt}{t}\Big)^{1/r} < \infty\Big\}, \quad (3.15)$$

if $0 < \theta < 1, 0 < r < \infty$, and

$$(X_0, X_2)_{\theta,\infty} := \Big\{a \in X_0 + X_1 : \|a\|_{(X_0,X_2)_{\theta,\infty}} := \sup_{0<t<\infty} t^{-\theta} K(t, a; X_0, X_1) < \infty\Big\}, \quad (3.16)$$

for $0 < \theta < 1$.

More details regarding the real method of interpolation can be found in [30].

Theorem 8 *([17] Theorem 5.1) Let $s_1, s_2 \in \mathbb{R},\ s_1 \neq s_2,\ 0 < r_1,\ r_2,\ r \leq \infty,\ 0 < \theta < 1,\ s = (1-\theta)s_1 + \theta s_2$. Then*

$$(F^{s_1}_{p,r_1}, F^{s_2}_{p,r_2})_{\theta,r} = B^{s}_{p,r}, \quad 0 < p < \infty, \quad (3.17)$$

$$(B^{s_1}_{p,r_1}, B^{s_2}_{p,r_2})_{\theta,r} = B^{s}_{p,r}, \quad 0 < p \leq \infty. \quad (3.18)$$

Furthermore, similar formulas hold for the homogeneous versions of the Besov and Triebel-Lizorkin spaces.

Now we start to prove Lemma 2. Owning to the arbitrariness of $r_i,\ i = 1,2$ in Theorem 8, let $r_1 = r_2 = r$, we have

$$(\dot{F}^{s_1}_{p,r}, \dot{F}^{s_2}_{p,r})_{\theta,r} = \dot{B}^{s}_{p,r}, \quad 0 < p < \infty. \tag{3.19}$$

Assume $\mathfrak{m} \in \dot{\mathfrak{m}}_{p,r}$ which equivalently says that

$$\|T_{\mathfrak{m}} f\|_{\dot{F}^{s}_{p,r}} \preceq \|f\|_{\dot{F}^{s}_{p,r}}, \quad \text{for any } f \in \dot{F}^{s}_{p,r}, \quad \text{with } s \in \mathbb{R}. \tag{3.20}$$

Next we only have to show that

$$\|T_{\mathfrak{m}} f\|_{\dot{B}^{s}_{p,r}} \preceq \|f\|_{\dot{B}^{s}_{p,r}}, \text{ for any } f \in \dot{B}^{s}_{p,r} \text{ with } \quad 0 < r \preceq \infty \quad \text{and} \quad s \in \mathbb{R}. \tag{3.21}$$

Indeed, if $f \in \dot{B}^{s}_{p,r}$, then $f = f_1 + f_2$ with $f_1 \in \dot{F}^{s_1}_{p,r},\ f_1 \in \dot{F}^{s_1}_{p,r}$. Thus $T_{\mathfrak{m}} f = T_{\mathfrak{m}} f_1 + T_{\mathfrak{m}} f_2$, (3.14) and (3.20) imply that

$$\begin{aligned} K(t, T_{\mathfrak{m}} f; \dot{F}^{s_1}_{p,r}, \dot{F}^{s_2}_{p,r}) &\preceq \inf(\|T_{\mathfrak{m}} f_1\|_{\dot{F}^{s_1}_{p,r}} + \|T_{\mathfrak{m}} f_2\|_{\dot{F}^{s_2}_{p,r}}) \\ &\preceq \inf(\|f_1\|_{\dot{F}^{s_1}_{p,r}} + \|f_2\|_{\dot{F}^{s_2}_{p,r}}). \end{aligned}$$

By a simple computation, we can prove that (3.21) holds, so the proof of Lemma 2 is finished.

3.3 Main results for the generalized heat equations

Now, we are in a position to state one of the main results for the generalized heat equations.

Theorem 9 *Assume that u is the solution to equation (3.1). Then for any $0 < p_*,\ p,\ q,\ r < \infty$ with $p_* \geq p$ and $\beta,\ \gamma \in \mathbb{R}$, we have the following estimate,*

$$\|u(t,\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{-\frac{n}{2\sigma}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2\sigma}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}} \qquad p_* > p, \tag{3.22}$$

and

$$\|u(t,\cdot)\|_{\dot{F}^{\beta}_{p,q}} \preceq \|u_0\|_{\dot{F}^{\beta}_{p,q}} \qquad 0 < p < \infty. \tag{3.23}$$

Proof. Let us begin with proving the first inequality (3.22). Note that, for any $\xi \in \mathbb{R}^n \backslash \{0\}$ and any nonnegative real number $\tilde{\beta}$ which will be determined later, we have the following identity

$$\begin{aligned} m(t,\xi) &= e^{-t|\xi|^{2\sigma}} (t|\xi|^{2\sigma})^{\tilde{\beta}} \frac{1}{(t|\xi|^{2\sigma})^{\tilde{\beta}}} && (3.24) \\ &= m_1(t,\xi) m_2(t,\xi), && (3.25) \end{aligned}$$

where

$$m_1(t,\xi) = e^{-t|\xi|^{2\sigma}} (t|\xi|^{2\sigma})^{\tilde{\beta}},$$

and

$$m_2(t,\xi) = \frac{1}{(t|\xi|^{2\sigma})^{\tilde{\beta}}}.$$

Clearly, $|m_1(t,\xi)| \le C$ holds uniformly with respect to ξ and $t \in (0,\infty)$, where C is a constant which depends on $\tilde{\beta}$. Thus, we can prove that $m_1(t,\xi)$ and $m_2(t,\xi)$ satisfy condition (2.18) with $\alpha = 0$ and $\alpha = 2\sigma\tilde{\beta}$ respectively. To be specific,

$$\sup_{R>0}\left[R^{-n+2|\tilde{\sigma}|}\int_{R<|\xi|<2R}|\partial_\xi^{\tilde{\sigma}} m_1(t,\xi)|^2 d\xi\right] \le A_{1,\tilde{\sigma}}, \quad |\tilde{\sigma}| \le l, \tag{3.26}$$

and

$$\sup_{R>0}\left[R^{-n+4\sigma\tilde{\beta}+2|\tilde{\sigma}|}\int_{R<|\xi|<2R}|\partial_\xi^{\tilde{\sigma}} m_2(t,\xi)|^2 d\xi\right] \le t^{-\tilde{\beta}} A_{2,\tilde{\sigma}}, \quad |\tilde{\sigma}| \le l, \tag{3.27}$$

where $A_{1,\tilde{\sigma}}$ and $A_{2,\tilde{\sigma}}$ are some constants.

Thus, combining Theorem 6 with $\alpha = 0$ and Corollary 1 with $\alpha = 2\sigma\tilde{\beta}$, we have

$$\begin{aligned}
\|u(t,\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} &= \|T_m u_0\|_{\dot{F}^{\beta}_{p_*,q}} \\
&= \|T_{m_1 m_2} u_0\|_{\dot{F}^{\beta}_{p_*,q}} \\
&= \|T_{m_1} T_{m_2} u_0\|_{\dot{F}^{\beta}_{p_*,q}} \\
&\preceq \|T_{m_2} u_0\|_{\dot{F}^{\beta}_{p_*,q}} \\
&\preceq t^{-\tilde{\beta}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}} \\
&= t^{-\frac{n}{2\sigma}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2\sigma}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}},
\end{aligned}$$

where $\tilde{\beta}$ is determined by $\beta - n/p_* = 2\sigma\tilde{\beta} + \gamma - n/p$, which derives our desired result (3.22).

To prove the second inequality in (3.23) in Theorem 9, we do not need to decompose the multiplier as in (3.24). Indeed, the inequality in (3.23) follows immediately from Theorem 6 with $\alpha = 0$. So we omit the details.

Since $\dot{F}^0_{p,2} = H^p$ $(0 < p < \infty)$, letting $\beta = \gamma = 0$ and $q = r = 2$ in Theorem (9), then the result is exactly the same as the known result in [3] which is stated as follows:

Theorem 10 *([3]) Assume $f \in H^r(\mathbb{R}^n)$. Then for $0 < r \le p \le \infty$,*

$$\left\|e^{-t(-\Delta)^\alpha} f\right\|_{H^p\mathbb{R}^n)} \preceq t^{-\frac{n}{2\alpha}(1/r-1/p)} \|f\|_{H^r\mathbb{R}^n)}. \tag{3.28}$$

The proof there easily yields

$$\left\|\partial_t^k \partial_x^\beta e^{-t(-\Delta)^\alpha} f\right\|_{H^p} \preceq t^{-\frac{n}{2\alpha}\{(1/r-1/p)+|\beta|\}-k} \|f\|_{H^r(\mathbb{R}^n)}.$$

Similarly, we can get the counterpart in Besov spaces.

Theorem 11 *Assume u is the solution to equation (3.1). Then for any* $0 < p_*,\ p,\ q,\ r < \infty$ *with* $p_* \geq p$ *and* $\beta,\ \gamma \in \mathbb{R}$*, we have the following estimate*

$$\|u(t,\cdot)\|_{\dot{B}^{\beta}_{p_*,q}} \preceq t^{-\frac{n}{2\sigma}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2\sigma}} \|u_0\|_{\dot{B}^{\gamma}_{p,r}} \qquad p_* > p, \tag{3.29}$$

and

$$\|u(t,\cdot)\|_{\dot{B}^{\beta}_{p,q}} \preceq \|u_0\|_{\dot{B}^{\beta}_{p,r}} \qquad 0 < p < \infty. \tag{3.30}$$

Proof. The proof is similar to the case in $\dot{F}^{\gamma}_{p,r}$ space. For any $\xi \in \mathbb{R}^n \setminus \{0\}$ and any nonnegative real number $\tilde{\beta}$ which will be determined later, we have the following identity

$$m(t,\xi) = e^{-t|\xi|^{2\sigma}}(t|\xi|^{2\sigma})^{\tilde{\beta}} \frac{1}{(t|\xi|^{2\sigma})^{\tilde{\beta}}} = m_1(t,\xi)m_2(t,\xi), \tag{3.31}$$

where $m_1(t,\xi) = e^{-t|\xi|^{2\sigma}}(t|\xi|^{2\sigma})^{\tilde{\beta}}$ and $m_2(t,\xi) = \frac{1}{(t|\xi|^{2\sigma})^{\tilde{\beta}}}$.

It is easy to prove that $m_1(t,\xi)$ and $m_2(t,\xi)$ satisfy condition (2.18) with $\alpha = 0$ and $\alpha = 2\sigma\tilde{\beta}$ respectively. To be specific,

$$\sup_{R>0}\left[R^{-n+2|\tilde{\sigma}|}\int_{R<|\xi|<2R} |\partial_{\xi}^{\tilde{\sigma}} m_1(t,\xi)|^2 d\xi\right] \leq A_{1,\tilde{\sigma}}, |\tilde{\sigma}| \leq l. \tag{3.32}$$

and

$$\sup_{R>0}\left[R^{-n+4\sigma\tilde{\beta}+2|\tilde{\sigma}|}\int_{R<|\xi|<2R} |\partial_{\xi}^{\tilde{\sigma}} m_2(t,\xi)|^2 d\xi\right] \leq t^{-\tilde{\beta}} A_{2,\tilde{\sigma}}, |\tilde{\sigma}| \leq l. \tag{3.33}$$

Combining Theorem 6 with $\alpha = 0$, Lemma 2 and Corollary 2 with $\alpha = 2\sigma\tilde{\beta}$, we obtain

$$\begin{aligned}
\|u(t,\cdot)\|_{\dot{B}^{\beta}_{p_*,q}} &= \|T_m u_0\|_{\dot{B}^{\beta}_{p_*,q}} \\
&= \|T_{m_1 m_2} u_0\|_{\dot{B}^{\beta}_{p_*,q}} \\
&= \|T_{m_1} T_{m_2} u_0\|_{\dot{B}^{\beta}_{p_*,q}} \\
&\preceq \|T_{m_2} u_0\|_{\dot{B}^{\beta}_{p_*,q}} \\
&\preceq t^{-\tilde{\beta}} \|u_0\|_{\dot{B}^{\gamma}_{p,r}} \\
&= t^{-\frac{n}{2\sigma}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2\sigma}} \|u_0\|_{\dot{B}^{\gamma}_{p,r}},
\end{aligned}$$

which proves inequality (3.29).

Inequality (3.30) can be derived directly with the application of the Theorem 6 and Lemma 2. We here skip the details.

Remark. Since $m_1 \in M_{p_*}$ and $M_{p_*} \subset \dot{\mathfrak{m}}_{p_*}$ for $1 < p_* < \infty$, it is not necessary to employ Theorem 6, Lemma 2 or Corollary 2 in the proof of Theorem 11.

Chapter 4

Wave Equations

In this central chapter, we shall study the long time decay estimates of the generalized wave equations in Triebel-Lizorkin spaces.

4.1 Introduction

We begin by studying the Cauchy problem for wave equations with a structure damping term,

$$\begin{cases} u_{tt} + 2a(-\Delta)^{\delta}u_t + (-\Delta)^{\sigma}u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), \end{cases} \tag{4.1}$$

where $0 < \delta < 1, a > 0$. The initial data u_0 and u_1 belong to some Triebel-Lizorkin spaces. The term $2a(-\Delta)^{\delta}u_t$, where$(-\Delta)^{\delta}f := \mathcal{F}^{-1}(|\xi|^{2\delta}\mathcal{F}f)$, $\mathcal{F}$ being the Fourier transform with respect to the spatial variable x, represents the action of a structural damping, which dissipates the energy of the solution to (4.1), as $t \to \infty$. In other words, the damping term $2a(-\Delta)^{\delta}u_t$ affects the behavior of the solution to equation (4.1).
First, for generalized wave equations without the damping term $2a(-\Delta)^{\delta}u_t$ are given by

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma}u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), \end{cases} \tag{4.2}$$

where $\sigma > 0$. The energy for (4.2) is given by (see [15])

$$E(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \|(-\Delta)^{\frac{\sigma}{2}}u(t,\cdot)\|_{L^2}^2, \tag{4.3}$$

and it remains constant for the solution to the Cauchy problem (4.2). Equivalently, $E^{'}(t) = 0$ for any $t \in [0,\infty)$.

Next let us verify that the damping term $2a(-\Delta)^{\delta}u_t$ does affect the behavior of the solution to equation (4.1). For simplicity, here we assume the solution $u(t,x)$ of equation

(4.2) is "good" enough, which means that we can avoid integrable problems, interchange integration and differentiation freely without further justification. The extension to general u in Triebel-Lizorkin spaces is done by a standard density argument. Thus, under our assumption, we have

$$\begin{aligned} E(t) &= \int_{\mathbb{R}^n} |u_t(x,t)|^2 dx + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\sigma}{2}} u(x,t)|^2 dx \\ &= \int_{\mathbb{R}^n} |\hat{u}_t(\xi,t)|^2 d\xi + \int_{\mathbb{R}^n} |\xi^\sigma \hat{u}(\xi,t)|^2 d\xi \quad \text{(by Plancherel's theorem)}. \end{aligned}$$

Then, taking the derivative with respect to the time t and with our assumption that the solution $u(t,x)$ is "good" enough, we get

$$\begin{aligned} E^{'}(t) &= 2\int_{\mathbb{R}^n} \hat{u}_t(\xi,t)\overline{\hat{u}_{tt}(\xi,t)} d\xi + \int_{\mathbb{R}^n} 2\xi^{2\sigma}\hat{u}(\xi,t)\overline{\hat{u}_t(\xi,t)} d\xi \\ &= 2\int_{\mathbb{R}^n} \hat{u}_t(\xi,t)\overline{(\hat{u}_{tt}(\xi,t) + \xi^{2\sigma}\hat{u}(\xi,t))} d\xi \\ &= 2\int_{\mathbb{R}^n} \hat{u}_t(\xi,t)\overline{(\hat{u}_{tt}(\xi,t) + \widehat{(-\Delta)^\sigma} u(\xi,t)} d\xi \\ &= 2\int_{\mathbb{R}^n} u_t(x,t)\overline{(u_{tt}(x,t) + (-\Delta)^\sigma u)} dx \\ &= 0, \end{aligned}$$

where, in the last step, we made use of the fact that $u(t,x)$ is the solution of equation (4.2).

However, the energy in (4.3) dissipates as $t \to \infty$ if we add a structural damping term $2a(-\Delta)^\delta u_t$, where $\delta \in (0,\sigma)$. Namely, the energy to the solution to the Cauchy problem (4.1) satisfies

$$E^{'}(t) = -4a\|(-\Delta)^{\frac{\delta}{2}} u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \leq 0.$$

In fact, taking the derivative with respect to the time t at both sides of the identity (4.3) and following the similar procedures as we did above, we get

$$\begin{aligned}
E^{'}(t) &= 2\int_{\mathbb{R}^n} u_t(x,t)\overline{(u_{tt}(x,t)+(-\Delta)^{\sigma}u(x,t))}dx \\
&= 2\int_{\mathbb{R}^n} u_t(x,t)\overline{(-2a(-\Delta)^{\delta}u_t(x,t))}dx \quad \text{(since } u(x,t) \text{ is the solution of (4.1))} \\
&= -4a\int_{\mathbb{R}^n} u_t(x,t)\overline{(-\Delta)^{\delta}u_t(x,t)}dx \\
&= -4a\int_{\mathbb{R}^n} \hat{u}_t(\xi,t)|\xi|^{2\delta}\overline{\hat{u}_t(\xi,t))}dx \quad \text{(by Plancherel's theorem)} \\
&= -4a\int_{\mathbb{R}^n} |\xi|^{\delta}\hat{u}_t(\xi,t)|\xi|^{\delta}\overline{\hat{u}_t(\xi,t)}d\xi \\
&= -4a\int_{\mathbb{R}^n} \Big||\xi|^{\delta}\hat{u}_t(\xi,t)\Big|^2 d\xi \\
&= -4a\int_{\mathbb{R}^n} \Big|(-\Delta)^{\frac{\delta}{2}}u_t(x,t)\Big|^2 dx \quad \text{(by Plancherel's theorem)} \\
&= -4a\|(-\Delta)^{\frac{\delta}{2}}u(t,x)\|^2_{L^2(\mathbb{R}^n)} \\
&\leq 0.
\end{aligned}$$

In the last step, the inequality holds because of the restriction on the parameter a with $a > 0$.

The solution to (4.1), according to the sign of $\sigma - 2\delta$, has different properties (see [18]). Also, we classify the damping term in equation (4.1) depending on the sign of $\sigma - 2\delta$. Namely, in equation (4.1), when $2\delta \in (0, \sigma)$, the damping is called effective, whereas it is called non effective when $\sigma \in (\delta, 2\delta)$. This classification was introduced by the first two authors [7] for more general models of evolution equations with time-dependent structural damping. For the details, see the paper [7] and the references therein. For the limit case $\delta = 1$ (see [14], [15], [27], [28]), and for $\sigma = 2\delta$, we refer the reader to [22]. Here we will not look at them in depth.

Also the partial differential equation (4.1) is significantly interesting in mathematics, physics, biology and many scientific fields. As known, the equation (4.1) has a lot of variants. For example, the classical wave equation, the classical heat equation, Laplace equation and Schrödinger equation which are fundamental types of partial differential equations. Let us finally mention that there are a lot of papers containing similar results on large time behavior of solutions to equations with structure dissipation. Our list of such papers is by no means exhaustive-we only cite the publications which have a direct influence on this paper. However, to my knowledge, there are quite few papers about applications to problems in partial differential equations, especially wave equation, in Triebel-Lizorkin Spaces. I just found that some literatures related to applications to the Laplacian and heat equations in Triebel-Lizorkin space ([16], [17]).

Without loss of generality, we may assume $a = 1$ in (4.1), since there exists a change of variables which makes coefficient to be unitary. Following the notation in [18], the solution to equation (4.1) is given by

$$u(t,\cdot) = S(t,\cdot) * u_0(\cdot) + T(t,\cdot) * u_1(\cdot). \tag{4.4}$$

In the expression above, the linear operators $S(t,\cdot)*$ and $T(t,\cdot)*$ are defined as the Fourier multiplier operators

$$\mathcal{F}(S(t,\cdot) * u_0(\cdot))(\xi) = m_S(t,\xi)\hat{u}_0(\xi), \tag{4.5}$$

and

$$\mathcal{F}(T(t,\cdot) * u_0(\cdot))(\xi) = m_T(t,\xi)\hat{u}_1(\xi), \tag{4.6}$$

with the symbols

$$m_S(t,\xi) = e^{-t|\xi|^{2\delta}} \left(\cosh\left(t\,|\xi|^{2\delta}\,\mu\right) + \frac{1}{\mu}\sinh\left(t\,|\xi|^{2\delta}\,\mu\right)\right), \tag{4.7}$$

and

$$m_T(t,\xi) = e^{-t|\xi|^{2\delta}} \frac{\sinh\left(t\,|\xi|^{2\delta}\,\mu\right)}{\mu\,|\xi|^{2\delta}}, \tag{4.8}$$

where

$$\begin{aligned} \mu &:= \mu(\xi) = \sqrt{1 - |\xi|^{2(\sigma-2\delta)}} \;\text{ if } |\xi|^{2(\sigma-2\delta)} < 1, \\ \mu &:= \mu(\xi) = i\sqrt{|\xi|^{2(\sigma-2\delta)} - 1} \;\text{ if } |\xi|^{2(\sigma-2\delta)} > 1. \end{aligned}$$

When $|\xi| = 1$, we may replace m_S and m_T by their limits as $|\xi| \to 1$, namely,

$$m_S(t,\xi)\,|_{|\xi|=1} = (1+t)e^{-t}, \qquad m_T(t,\xi)\,|_{|\xi|=1} = te^{-t}.$$

4.2 Some known results in real Hardy spaces

Before we formulate our first main proposition, let us recall the main results in real Hardy spaces [6].

Theorem 12 *([6]) Let $2\delta \in (0,\sigma)$. Then the solution u to (4.1) satisfies the decay estimate*

$$\left\|\partial_t^k \partial_x^\alpha u(t,\cdot)\right\|_{H^q} \preceq t^{-\frac{1}{2(\sigma-\delta)}\{(n(1/p-1/q)+|\alpha|\}-k} \left(\|u_0\|_{H^p} + \|u_1\|_{H^r}\right)$$

for $p \in (0,1]$, $p \le q \le \infty$, $t > 1$ and $1/r = 1/p + \frac{2\delta}{n}$.

Theorem 13 *([6]) Let $\sigma \in (\delta, 2\delta)$. Then the solution u to (4.1) satisfies the estimate*

$$\left\|\partial_t^k \partial_x^\alpha u(t,\cdot)\right\|_{H^q} \preceq t^{n(\frac{1}{q}-\frac{1}{2})_+ + (1-\frac{\sigma}{2\delta}) - \frac{1}{2\delta}\left(n(\frac{1}{p}-\frac{1}{q})+|\alpha|+k\sigma\right)} \left(\|u_0\|_{H^p} + t^{-(1-\frac{\sigma}{2\delta})}\|u_1\|_{H^r}\right)$$

for $p \in (0,1]$, $p \le q \le \infty$, $t > 1$ and $1/r = 1/p + \frac{2\delta}{n}$,

where $(x)_+ = max\{x, 0\}$, the positive part of x.

Since $\dot{F}^0_{p_*,2} = H^{p^*}$, our main goal is to extend the results in real Hardy space [6] to the homogeneous Triebel-Lizorkin space (the homogeneous Besov spaces). However, due to the lack of the corresponding multiplier Theorem 4 [25], we could not apply the known results in [3] or multiplier theorem directly, at least for some indices in $\dot{F}^\beta_{p_*,q}$, for the non-effective case. For this reason, it is convenient to distinguish the several cases $1 < p_* \le q \le 2$, $0 < p_* \le q \le 2$ with $p_* < 1$, and $2 \le q \le p_* < \infty$, where the interpolation and duality methods play a crucial role.

4.3 Main results for the generalized wave equations

After these preliminaries, we are in a position to present our first main result in this paper. Based on the sign of $\sigma - 2\delta$, we consider the first situation which is effective case $\sigma - 2\delta > 0$. Thanks to Theorem 10, we just need to estimate the simplest case of solution of equation (4.1) without the derivatives.

Case I effective case: $\sigma - 2\delta > 0$.

Proposition 3 *Suppose $\sigma > 2\delta > 0$, and $u(t,\cdot)$ is the solution of (4.1). We have*

$$\|u(t,\cdot)\|_{\dot{F}^\beta_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)} t^{-\frac{\beta-r}{2(\sigma-\delta)}} \left(\|u_0\|_{\dot{F}^\gamma_{p,r}} + \|u_1\|_{\dot{F}^{\tilde{\gamma}}_{\tilde{p},r}}\right) \tag{4.9}$$

for $0 < p \le p_* \le \infty$, $\frac{1}{\tilde{p}} = \frac{2\delta+\tilde{\gamma}-\gamma}{n} + \frac{1}{p}$, and $t > 1$.

We also have

$$\|u(t,\cdot)\|_{\dot{F}^\beta_{p_*,q}} \preceq \|u_0\|_{\dot{F}^\beta_{p_*,q}}, \quad p = p_*. \tag{4.10}$$

To prove inequality (4.9), since $u(t,\cdot) = S(t,\cdot) * u_0(\cdot) + T(t,\cdot) * u_1(\cdot)$, it suffices to prove the following two inequalities respectively

$$\|S(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)} t^{-\frac{\beta-r}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}}, \tag{4.11}$$

and

$$\|T(t,\cdot) * u_1(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)} t^{-\frac{\beta-r}{2(\sigma-\delta)}} \|u_1\|_{\dot{F}^{\tilde{\gamma}}_{\tilde{p},r}}. \tag{4.12}$$

First, we prove (4.11) by dividing its proof into several lemmas. Recall that

$$S(t,\cdot) * u_0(\cdot) = S_c(t,\cdot) * u_0(\cdot) + S_s(t,\cdot) * u_0(\cdot) \tag{4.13}$$

where $S_c(t,\cdot)*$ has the symbol $e^{-t|\xi|^{2\delta}}\left(\cosh\left(t\left|\xi\right|^{2\delta}\mu\right)\right)$ and $S_s(t,\cdot)*$ has the symbol $e^{-t|\xi|^{2\delta}}\frac{1}{\mu}\sinh\left(\left(t\left|\xi\right|^{2\delta}\mu\right)\right.$. Also we know

$$m_T(t,\xi)_{|\xi|=1} = te^{-t},\ m_S(t,\xi)_{|\xi|=1} = (1+t)e^{-t}.$$

Proof of (4.11). The proof consists in finding appropriate estimates of $S_c(t,\cdot) * u_0(\cdot)$ and $S_s(t,\cdot) * u_0(\cdot)$ in (4.13). To show that equality (4.11) holds, it is sufficient to prove that

$$\|S_c(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)} t^{-\frac{\beta-r}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}}, \tag{4.14}$$

and

$$\|S_s(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)} t^{-\frac{\beta-r}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}}. \tag{4.15}$$

To apply partition of unit, let $\Phi_0, \Phi_1, \Phi_\infty$ be cut-off nonnegative C^∞ functions satisfying

$$\begin{aligned} \Phi_0 + \Phi_1 + \Phi_\infty &\equiv 1,\ \text{supp}\ \Phi_0 \subset \{|\xi| \le 1/2\}, \\ \text{supp}\ \Phi_1 &\subset \{1/4 \le |\xi| \le 4\},\ \text{supp}\ \Phi_\infty \subset \{|\xi| \ge 2\}. \end{aligned}$$

Notice that

$$\begin{aligned} & e^{-t|\xi|^{2\delta}}\cosh\left(t\left|\xi\right|^{2\delta}\mu\right) \\ = \ & e^{-t|\xi|^{2\delta}}\frac{e^{t|\xi|^{2\delta}\mu} + e^{-t|\xi|^{2\delta}\mu}}{2}. \end{aligned}$$

For simplicity, we may assume that the symbol of $S_c(t,\cdot)*$ is $e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu}$.

Now we write

$$e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu} = e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu}\left(\Phi_0(\xi)+\Phi_1(\xi)+\Phi_\infty(\xi)\right).$$

By this decomposition, we have

$$S_c(t,\cdot)*u_0(\cdot) = S_{c,0}(t,\cdot)*u_0(\cdot)+S_{c,1}(t,\cdot)*u_0(\cdot)+S_{c,\infty}(t,\cdot)*u_0(\cdot) \tag{4.16}$$

where for $j=0,1,\infty$, $S_{c,j}(t,\cdot)*$ has the symbol

$$e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu}\Phi_j(\xi).$$

To get inequality (4.14), we need to prove that

$$\|S_{c,j}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)}t^{-\frac{\beta-r}{2(\sigma-\delta)}}\|u_0\|_{\dot{F}^{\gamma}_{p,r}},$$

for $j=0,1,\infty$.

Estimate of $S_{c,0}(t,\cdot)*u_0(\cdot)$. We start with the estimate of the first term on the right hand side of (4.16). Note that for $|\xi|\leq 1/2$, we may write

$$\begin{aligned}
&\sqrt{1-|\xi|^{2(\sigma-2\delta)}}\\
=\;& 1-\frac{|\xi|^{2(\sigma-2\delta)}}{2}+\sqrt{1-|\xi|^{2(\sigma-2\delta)}}-1+\frac{|\xi|^{2(\sigma-2\delta)}}{2}\\
=\;& 1-\frac{|\xi|^{2(\sigma-2\delta)}}{2}-g(\xi),
\end{aligned}$$

where

$$g(\xi) = 1-\frac{|\xi|^{2(\sigma-2\delta)}}{2}-\sqrt{1-|\xi|^{2(\sigma-2\delta)}}. \tag{4.17}$$

By the Taylor expansion, we have

$$g(\xi) = \frac{1}{8}\left(|\xi|^{2(\sigma-2\delta)}\right)^2+\frac{1}{16}\left(|\xi|^{2(\sigma-2\delta)}\right)^3+\frac{5}{128}\left(|\xi|^{2(\sigma-2\delta)}\right)^4+\cdots. \tag{4.18}$$

From the identity in (4.18), it follows that $g(\xi)>0$ for any $\xi\neq 0$. Therefore, we have that, for $|\xi|\leq 1/2$,

$$\begin{aligned}
e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu} &= \exp -t\,|\xi|^{2\delta}\exp t\,|\xi|^{2\delta}\left(1-\frac{|\xi|^{2(\sigma-2\delta)}}{2}-g(\xi)\right)\\
&= \exp -t\,|\xi|^{2\delta}+t\,|\xi|^{2\delta}\left(1-\frac{|\xi|^{2(\sigma-2\delta)}}{2}-g(\xi)\right).
\end{aligned}$$

Here, the exponent is

$$
\begin{aligned}
& -t\,|\xi|^{2\delta} + t\,|\xi|^{2\delta}\left(1 - \frac{|\xi|^{2(\sigma-2\delta)}}{2} - g(\xi)\right) \\
= & -\frac{t\,|\xi|^{2\delta}\,|\xi|^{2(\sigma-2\delta)}}{2} - t\,|\xi|^{2\delta}\,g(\xi) \\
= & -\frac{t\,|\xi|^{2(\sigma-\delta)}}{2} - t\,|\xi|^{2\delta}\,g(\xi).
\end{aligned}
$$

Thus, we obtain that the symbol of $S_{c,0}(t,\cdot)*$ is

$$
\begin{aligned}
& \tilde{m}_1(t,\xi) = e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}\Phi_0(\xi) \\
= & \; e^{-\frac{t|\xi|^{2(\sigma-\delta)}}{2}} m_{1,1}(t,\xi) \\
= & \; e^{-\frac{t|\xi|^{2(\sigma-\delta)}}{2}} (t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta_1}} \frac{1}{(t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta_1}}} m_{1,1}(t,\xi) \\
= & \; m_{1,1}(t,\xi) m_{1,2}(t,\xi) m_{1,3}(t,\xi),
\end{aligned}
$$

where

$$
\begin{aligned}
m_{1,1}(t,\xi) &= e^{-t|\xi|^{2\delta}g(\xi)}\Phi_0(\xi), \\
m_{1,2}(t,\xi) &= e^{-\frac{t|\xi|^{2(\sigma-\delta)}}{2}} (t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta_1}},
\end{aligned}
$$

and

$$
m_{1,3}(t,\xi) = \frac{1}{(t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta_1}}}.
$$

To apply the Mikhlin-Hörmander multiplier theorem, we will establish the following lemma.

Lemma 3 *For any multi-index α,*

$$
\left|\frac{\partial^\alpha}{\partial\xi^\alpha} m_{1,1}(t,\xi)\right| \le C\,|\xi|^{-|\alpha|},
$$

where the constant C is independent of t.

Proof. Following the definition of multi-index derivative and employing the property of $\Phi_0(\xi)$, we can easily get

$$
\left|\frac{\partial^\alpha}{\partial\xi^\alpha} m_{1,1}(t,\xi)\right| \le C|v_1(t,\xi)|\frac{1}{|\xi|^{|\alpha|-|\beta|}},
$$

where the constant C depends only on δ, σ and α, but does not depend on t, $v_1(t,\xi)$ has the following form

$$
v_1(t,\xi) = e^{-t|\xi|^{2\delta}g(\xi)}(t|\xi|^{2\delta})^i \frac{\partial^{\beta_1}}{\partial\xi^{\beta_1}}g(\xi)\frac{\partial^{\beta_2}}{\partial\xi^{\beta_2}}g(\xi)\cdots\frac{\partial^{\beta_l}}{\partial\xi^{\beta_l}}g(\xi)g^j(\xi)\varphi_\alpha(\xi),
$$

and

$$0 \le i \le |\alpha|,\ 0 \le j \le |\alpha|,\ 0 \le l \le |\alpha|,$$

$$0 \le |\beta_k| \le |\alpha|,\ k = 1, 2, \cdots l,\ l + j = i.$$

$\varphi_\alpha(\xi)$ is bounded uniformly with respect to ξ.

Note that

$$\begin{aligned} g(\xi) > 0, \forall \xi \in \mathbb{R}^n \setminus \{0\}, \\ \Big|\frac{\partial^\alpha}{\partial \xi^\alpha} g(\xi)\Big| \preceq \frac{g(\xi)}{|\xi|^\alpha}. \end{aligned} \tag{4.19}$$

Thus, we have

$$\begin{aligned} &\Big|\frac{\partial^\alpha}{\partial \xi^\alpha} m_{1,1}(t,\xi)\Big| \\ &\le C|v_1(t,\xi)|\frac{1}{|\xi|^{|\alpha|-|\beta|}} \\ &\le Ce^{-t|\xi|^{2\delta}g(\xi)}(t|\xi|^{2\delta})^i \frac{g^l(\xi)}{|\xi|^{|\beta_1|+|\beta_2|+\cdots|\beta_l|}} g^j(\xi)\frac{1}{|\xi|^{|\alpha|-|\beta|}} \\ &= Ce^{-t|\xi|^{2\delta}g(\xi)}(t|\xi|^{2\delta})^i g^{(l+j)}(\xi)\frac{1}{|\xi|^{|\alpha|}} \\ &= Ce^{-t|\xi|^{2\delta}g(\xi)}(tg(\xi)|\xi|^{2\delta})^i \frac{1}{|\xi|^{|\alpha|}} \\ &\le C\frac{1}{|\xi|^{|\alpha|}}, \end{aligned} \tag{4.20}$$

where, in the last step, we employed the fact $|e^{-t|\xi|^{2\delta}g(\xi)}(tg(\xi)|\xi|^{2\delta})^i| \le C$ uniformly with respect to $\xi, t \in (0,\infty)$ and any non-negative integer i. So Lemma 3 is proved.
By Lemma 3, we know that $m_{1,1}(t,\xi)$ satisfies condition (2.18). Also, we may verify that $m_{1,2}(t,\xi)$ and $m_{1,3}(t,\xi)$ also satisfy (2.18) with $\alpha = 0$ and $\alpha = 2(\sigma-\delta)\tilde{\beta}_1$ respectively, where $\tilde{\beta}_1$ is a nonnegative real number which will be determined later on.

Thus,

$$\begin{aligned} &\|S_{c,0}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}} \\ &= \|T_{m_1} u_0\|_{\dot{F}^\beta_{p_*,q}} \\ &= \|T_{m_{1,1}} T_{m_{1,2}} T_{m_{1,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\ &\preceq \|T_{m_{1,2}} T_{m_{1,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \quad \text{(by Theorem 6 with } \alpha = 0) \\ &\preceq \|T_{m_{1,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \quad \text{(by Theorem 6 with } \alpha = 0) \\ &\preceq t^{-\tilde{\beta}_1} \|u_0\|_{\dot{F}^\gamma_{p,r}} \quad \text{(by Corollary 1)} \\ &= t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^\gamma_{p,r}} \quad p_* > p, \end{aligned} \tag{4.21}$$

where $\tilde{\beta_1}$ is determined by $\beta - n/p_* = 2(\sigma-\delta)\tilde{\beta_1} + \gamma - n/p$, which derives our desired result, obtaining the estimate of $S_{c,0}(t,\cdot) * u_0(\cdot)$.

Estimate of $S_{c,\infty}(t,\cdot) * u_0(\cdot)$. Recall that the symbol of $S_{c,\infty}(t,\cdot)*$ is

$$\begin{aligned} & e^{-t|\xi|^{2\delta}} \cosh\left(it\,|\xi|^{2\delta}\sqrt{|\xi|^{2(\sigma-2\delta)}-1}\right)\Phi_\infty(\xi) \\ = \;& \Phi_\infty(\xi)\, e^{-t|\xi|^{2\delta}} \frac{e^{it|\xi|^{2\delta}\sqrt{|\xi|^{2(\sigma-2\delta)}-1}} + e^{-it|\xi|^{2\delta}\sqrt{|\xi|^{2(\sigma-2\delta)}-1}}}{2}. \end{aligned}$$

We proceed the same as before, it suffices to study the symbol

$$e^{-t|\xi|^{2\delta}} e^{it|\xi|^{2\delta}\sqrt{|\xi|^{2(\sigma-2\delta)}-1}}\Phi_\infty(\xi).$$

We write

$$\sqrt{|\xi|^{2(\sigma-2\delta)}-1} = |\xi|^{(\sigma-2\delta)}\sqrt{1-|\xi|^{-2(\sigma-2\delta)}}.$$

Therefore the symbol of $S_{c,\infty}(t,\cdot)*$ is

$$\begin{aligned} \tilde{m}_2(t,\xi) &= e^{-t|\xi|^{2\delta}} e^{it|\xi|^{2\delta}|\xi|^{(\sigma-2\delta)}\sqrt{1-|\xi|^{-2(\sigma-2\delta)}}}\Phi_\infty(\xi) \\ &= e^{-t|\xi|^{2\delta}} e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}\Phi_\infty(\xi) \\ &= e^{-\frac{t}{2}|\xi|^{2\delta}}(t|\xi|^{2\delta})^{\tilde{\beta_2}}\frac{1}{(t|\xi|^{2\delta})^{\tilde{\beta_2}}} m_{2,1}(t,\xi) \\ &= m_{2,1}(t,\xi)m_{2,2}(t,\xi)m_{2,3}(t,\xi), \end{aligned}$$

where

$$\begin{aligned} m_{2,1}(t,\xi) &= e^{-\frac{t}{2}|\xi|^{2\delta}} e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}\Phi_\infty(\xi) \\ &= e^{-\frac{t}{2}|\xi|^{2\delta}} e^{it|\xi|^{\sigma}h(\xi)}\Phi_\infty(\xi), \end{aligned}$$

$$m_{2,2}(t,\xi) = e^{-\frac{t}{2}|\xi|^{2\delta}}(t|\xi|^{2\delta})^{\tilde{\beta_2}},$$

and

$$m_{2,3}(t,\xi) = \frac{1}{(t|\xi|^{2\delta})^{\tilde{\beta_2}}},$$

where $\tilde{\beta_2}$ is a nonnegative real number which will be determined later on and $h(\xi) = \sqrt{1-|\xi|^{2(2\delta-\sigma)}}$, $\xi \in \text{supp}\ \Phi_\infty$.
To proceed, we also need to prove the following lemma.

Lemma 4 *For any multi-index* α,

$$\left|\frac{\partial^\alpha}{\partial\xi^\alpha} m_{2,1}(t,\xi)\right| \le C\,|\xi|^{-|\alpha|},$$

where the constant C *is independent of* t.

Proof. Following the definition of multi-index derivative and employing the property of Φ_∞, we have

$$\Big|\frac{\partial^\alpha}{\partial\xi^\alpha} m_{2,1}(t,\xi)\Big| \le C|v_2(t,\xi)|\frac{1}{|\xi|^{|\alpha|-|\beta|}},$$

where the constant C depends only on δ, σ and α, but does not depend on t, $v_2(t,\xi)$ has the following form

$$v_2(t,\xi) = e^{-\frac{t}{2}|\xi|^{2\delta}}(t|\xi|^{2\delta})^i(|\xi|^{\sigma-2\delta})^l \frac{\partial^{\beta_1}}{\partial\xi^{\beta_1}}h(\xi)\frac{\partial^{\beta_2}}{\partial\xi^{\beta_2}}h(\xi)\cdots\frac{\partial^{\beta_r}}{\partial\xi^{\beta_r}}h(\xi)h^j(\xi)\omega_\alpha(t,\xi),$$

and

$$0 \le i \le |\alpha|,\ 0 \le j \le |\alpha|,\ 0 \le l \le |\alpha|,$$

$$0 \le |\beta_k| \le |\alpha|,\ k = 1, 2, \cdots r,\ r + j = l,$$

$\omega_\alpha(t,\xi)$ is bounded uniformly with respect to t and ξ.

Thus, we obtain that

$$\begin{aligned}
\Big|\frac{\partial^\alpha}{\partial\xi^\alpha} m_{2,1}(t,\xi)\Big| &\le C|v_2(t,\xi)|\frac{1}{|\xi|^{|\alpha|-|\beta|}} \\
&\le Ce^{-\frac{t}{2}|\xi|^{2\delta}}(\frac{t}{2}|\xi|^{2\delta})^i(|\xi|^{\sigma-2\delta})^l\frac{h^r(\xi)}{|\xi|^{|\beta_1|+|\beta_2|+\cdots|\beta_l|}}h^j(\xi)\frac{1}{|\xi|^{|\alpha|-|\beta|}} \\
&= Ce^{-\frac{t}{2}|\xi|^{2\delta}}(\frac{t}{2}|\xi|^{2\delta})^i(|\xi|^{\sigma-2\delta})^l h^{(r+j)}(\xi)\frac{1}{|\xi|^{|\alpha|}} \\
&= Ce^{-\frac{t}{2}|\xi|^{2\delta}}(\frac{t}{2}|\xi|^{2\delta})^i(|\xi|^{\sigma-2\delta})^l h^l(\xi)\frac{1}{|\xi|^{|\alpha|}} \\
&\le C(|\xi|^{2\delta-\sigma})^l\frac{1}{|\xi|^{|\alpha|}} \\
&\le C\frac{1}{|\xi|^{|\alpha|}},
\end{aligned}$$

where we employed the facts that $h(\xi) > 0, \forall\ \xi \in \text{supp}\ \Phi_\infty \subset \{|\xi| \ge 2\}$,

$$\Big|\frac{\partial^\alpha}{\partial\xi^\alpha}h(\xi)\Big| \preceq \frac{h(\xi)}{|\xi|^\alpha},$$

and

$$e^{-\frac{t}{2}|\xi|^{2\delta}}(\frac{t}{2}|\xi|^{2\delta})^i \le C, \qquad \text{for any nonnegative integer i.}$$

So we end the proof of Lemma 4 by virtue of the assumption $\sigma > 2\delta$.

Lemma 4 tells us that $m_{2,1}$ satisfies the condition (2.18) with $\alpha = 0$. Similarly, we also may prove that $m_{2,2}$ and $m_{2,3}$ satisfy the condition (2.18) with $\alpha = 0$ and $\alpha = 2\delta\tilde{\beta}_2$ respectively. Employing Theorem 6 and Corollary 1, we obtain that, for $\sigma - 2\delta > 0$,

$$\begin{aligned}
\|S_{c,\infty}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} &= \|T_{m_2} u_0\|_{\dot{F}^{\beta}_{p_*,q}} \\
&= \|T_{m_{2,1}} T_{m_{2,2}} T_{m_{2,3}} u_0\|_{\dot{F}^{\beta}_{p_*,q}} \\
&\preceq \|T_{m_{2,2}} T_{m_{2,3}} u_0\|_{\dot{F}^{\beta}_{p_*,q}} \qquad (\text{by Theorem 6 with } \alpha = 0) \\
&\preceq \|T_{m_{2,3}} u_0\|_{\dot{F}^{\beta}_{p_*,q}} \qquad (\text{by Theorem 6 with } \alpha = 0) \\
&\preceq t^{-\tilde{\beta}_2} \|u_0\|_{\dot{F}^{\gamma}_{p,r}} \qquad (\text{by Corollary 1}) \\
&= t^{-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2\delta}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}} \\
&\preceq t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}} \quad p_* > p,
\end{aligned} \tag{4.22}$$

where $\tilde{\beta}_2$ is determined by $\beta - n/p_* = 2\delta\tilde{\beta}_2 + \gamma - n/p$, which derives our desired result.

Remark. We can choose $\tilde{\beta}_2$ large enough , since the arbitrariness of $\tilde{\beta}_2$, such that $\beta > \gamma$. Then we obtain better decay estimate since $\sigma > 2\delta$.

Estimate of $\|S_{c,1}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}}$. It should be the easiest part. In particular, taking the advantage of the boundedness of *supp* $\Phi_1 \subset \{1/4 \le |\xi| \le 4\}$ and the smoothness of Φ_1, we proceed the same way as we did in Lemma 3 and Lemma 4, then apply Theorem 6 and Corollary 1, it is not too difficult to obtain the boundedness of $\|S_{c,1}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}}$ such that

$$\begin{aligned}
\|S_{c,1}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} &\preceq t^{-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2\delta}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}} \\
&\preceq t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}} \quad p_* > p.
\end{aligned} \tag{4.23}$$

Combining (4.21)-(4.23), we finish the proof of (4.14).

Remark. For the multiplier of $S_{c,1}(t,\cdot)*$, it seems that there is a singularity at $|\xi| = 1$ because of the denominator μ. Indeed we do not have to worry about it at all. Since

$$\begin{aligned}
m_S(t,\xi) &= e^{-t|\xi|^{2\delta}} \left(\cosh\left(t|\xi|^{2\delta}\mu\right) + \frac{1}{\mu}\sinh\left(t|\xi|^{2\delta}\mu\right) \right) \\
&= e^{-t|\xi|^{2\delta}} \cosh\left(t|\xi|^{2\delta}\mu\right) + e^{-t|\xi|^{2\delta}} \frac{1}{\mu}\sinh\left(t|\xi|^{2\delta}\mu\right)
\end{aligned}$$

employing the Taylor expansion of $\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ and $\sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$, we get

$$\begin{aligned}
m_S(t,\xi) &= e^{-t|\xi|^{2\delta}} \sum_{k=0}^{\infty} \frac{(t|\xi|^{2\delta}\mu)^{2k}}{(2k)!} + e^{-t|\xi|^{2\delta}} \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{(t|\xi|^{2\delta}\mu)^{2k+1}}{(2k+1)!} \\
&= e^{-t|\xi|^{2\delta}} \sum_{k=0}^{\infty} \frac{(t|\xi|^{2\delta}\mu)^{2k}}{(2k)!} + e^{-t|\xi|^{2\delta}} \sum_{k=0}^{\infty} \frac{(t|\xi|)^{2\delta})^{2k+1}\mu^{2k}}{(2k+1)!}.
\end{aligned}$$

Similarly, to show inequality (4.15) holds, we estimate the boundedness of $\|S_{s,i}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p*,q}}$ term by term, where $i = 0, 1, \infty$.
Notice that

$$\begin{aligned} &\frac{e^{-t|\xi|^{2\delta}} \sinh\left(t\,|\xi|^{2\delta}\mu\right)}{\mu} \\ =\ & e^{-t|\xi|^{2\delta}} \frac{e^{t|\xi|^{2\delta}\mu} + e^{-t|\xi|^{2\delta}\mu}}{2\mu}. \end{aligned}$$

For simplicity, we may assume the symbol of $S_c(t,x)*$ is

$$\frac{e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu}}{\mu}.$$

Now we may write

$$\frac{e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu}}{\mu} = \frac{e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu}}{\mu} \left(\Phi_0(\xi) + \Phi_1(\xi) + \Phi_\infty(\xi)\right).$$

By this decomposition, we have

$$S_s(t,\cdot) * u_0(\cdot) = S_{s,0}(t,\cdot) * u_0(\cdot) + S_{s,1}(t,\cdot) * u_0(\cdot) + S_{s,\infty}(t,\cdot) * u_0(\cdot), \tag{4.24}$$

where for $j = 0, 1, \infty$, $S_{s,j}(t,\cdot)*$ has the symbol

$$\frac{e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu}\Phi_j(\xi)}{\mu}.$$

Same as before, we now need to prove that

$$\|S_{s,j}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}}$$

for $j = 0, 1, \infty$.

Estimate of $S_{s,0}(t,\cdot) * u_0(\cdot)$. For $|\xi| \le 1/2$, we write

$$\sqrt{1 - |\xi|^{2(\sigma-2\delta)}} = 1 - \frac{|\xi|^{2(\sigma-2\delta)}}{2} - g(\xi),$$

where $g(\xi)$ is the same as (4.17) which means

$$g(\xi) = 1 - \frac{|\xi|^{2(\sigma-2\delta)}}{2} - \sqrt{1 - |\xi|^{2(\sigma-2\delta)}},$$

and $g(\xi) > 0$ for any $\xi \neq 0$. Therefore, we have that for $|\xi| \leq 1/2$,

$$\begin{aligned} e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu} &= \exp -t\,|\xi|^{2\delta} \exp t\,|\xi|^{2\delta}\left(1 - \frac{|\xi|^{2(\sigma-2\delta)}}{2} - g(\xi)\right) \\ &= \exp -t\,|\xi|^{2\delta} + t\,|\xi|^{2\delta}\left(1 - \frac{|\xi|^{2(\sigma-2\delta)}}{2} - g(\xi)\right). \end{aligned}$$

Here, the exponent is

$$\begin{aligned} & -t\,|\xi|^{2\delta} + t\,|\xi|^{2\delta}\left(1 - \frac{|\xi|^{2(\sigma-2\delta)}}{2} - g(\xi)\right) \\ = & -\frac{t\,|\xi|^{2\delta}\,|\xi|^{2(\sigma-2\delta)}}{2} - t\,|\xi|^{2\delta}\,g(\xi) \\ = & -\frac{t\,|\xi|^{2(\sigma-\delta)}}{2} - t\,|\xi|^{2\delta}\,g(\xi). \end{aligned}$$

Thus, we obtain that the symbol of $S_{s,0}(t,x)*$ is

$$\begin{aligned} m_3(t,\xi) &= \frac{e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}\Phi_0(\xi)}{\mu} \\ = \; & e^{-\frac{t|\xi|^{2(\sigma-\delta)}}{2}} (t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta}_3} \frac{1}{(t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta}_3}} m_{3,1}(t,\xi) \\ = \; & m_{3,1}(t,\xi) m_{3,2}(t,\xi) m_{3,3}(t,\xi) \end{aligned}$$

where

$$\begin{aligned} m_{3,1}(t,\xi) &= \frac{e^{-t|\xi|^{2\delta}g(\xi)}\Phi_0(|\xi|)}{\mu} \\ &= \frac{m_{1,1}(t,\xi)}{\mu}, \end{aligned}$$

$$m_{3,2}(t,\xi) = e^{-\frac{t}{2}|\xi|^{2(\sigma-\delta)}} (t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta}_3},$$

$$m_{3,3}(t,\xi) = \frac{1}{(t|\xi|^{2(\sigma-\delta)})^{\tilde{\beta}_3}},$$

and $\tilde{\beta}_3$ is a nonnegative real number which will be determined later on. Since we have the following estimates for $\sigma - 2\delta > 0$,

$$\left|\frac{\partial^\beta}{\partial \xi^\beta} m_{1,1}(t,\xi)\right| \leq C\,|\xi|^{-|\beta|},$$

and

$$\left|\frac{\partial^\gamma}{\partial \xi^\gamma}\frac{1}{\mu(t,\xi)}\right| \leq C\,|\xi|^{-|\gamma|}.$$

Leibniz's formula

$$\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g),$$

yields that

$$\left|\frac{\partial^\alpha}{\partial\xi^\alpha} m_{3,1}(t,\xi)\right| \le C\,|\xi|^{-|\alpha|}.$$

All above estimates imply that $m_{3,1}(t,\xi), m_{3,2}(t,\xi), m_{3,3}(t,\xi)$ satisfy condition (2.18) with $\alpha = 0,\ \alpha = 0$ and $\alpha = 2(\sigma-\delta)\tilde{\beta}_3$, separately. Combining Theorem 6 and Corollary 1, then we have

$$\begin{aligned}
\|S_{s,0}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}} &= \|T_{m_3} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&= \|T_{m_{3,1}} T_{m_{3,2}} T_{m_{3,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&\preceq \|T_{m_{3,2}} T_{m_{3,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \qquad (\text{by Theorem 6 with } \alpha = 0) \\
&\preceq \|T_{m_{3,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \qquad (\text{by Theorem 6 with } \alpha = 0) \\
&\preceq t^{-\tilde{\beta}_3} \|u_0\|_{\dot{F}^\gamma_{p,r}} \qquad (\text{by Corollary 1}) \\
&= t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^\gamma_{p,r}} \quad p_* > p,
\end{aligned} \tag{4.25}$$

where $\tilde{\beta}_3$ is determined by $\beta - n/p_* = 2(\sigma-\delta)\tilde{\beta}_1 + \gamma - n/p$.
Following the proof of Lemma 2.2, we can get the estimate

$$\|S_{s,\infty}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^\gamma_{p,r}} \quad p_* > p. \tag{4.26}$$

For the same reason as the boundedness of $\|S_{c,1}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}}$, we can get the following estimate

$$\|S_{s,1}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}} \preceq t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^\gamma_{p,r}}, \quad p_* > p. \tag{4.27}$$

By the decomposition (4.24), combining (4.25)-(4.27), we end the proof of (4.15).

We now come to the proof of the second main equality (4.12). Accordingly, we shall give both the low and high frequency estimates.

Proof of (4.12). Similarly, by partition of unit, we can rewrite the symbol of $T(t,\cdot)*$ as

$$\frac{e^{-t|\xi|^{2\delta}} \sinh\left(t\,|\xi|^{2\delta}\mu\right)}{|\xi|^{2\delta}\mu}\left(\Phi_0(\xi) + \Phi_1(\xi) + \Phi_\infty(\xi)\right) = m_{T0} + m_{T1} + m_{T\infty},$$

where $m_{Tj} = \frac{e^{-t|\xi|^{2\delta}}\sinh\left(t|\xi|^{2\delta}\mu\right)}{|\xi|^{2\delta}\mu}\Phi_j,\ j = 0, 1, \infty.$

First, we consider the first part of symbol of $T(x,t)*$

$$\begin{aligned} m_{T0} &= \frac{e^{-t|\xi|^{2\delta}}\sinh\left(t\left|\xi\right|^{2\delta}\sqrt{1-\left|\xi\right|^{2(\sigma-2\delta)}}\right)}{\left|\xi\right|^{2\delta}\sqrt{1-\left|\xi\right|^{2(\sigma-2\delta)}}}\Phi_0(\xi) \\ &= e^{-t|\xi|^{2\delta}}\frac{e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}-e^{-t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}}{2\left|\xi\right|^{2\delta}\sqrt{1-\left|\xi\right|^{2(\sigma-2\delta)}}}\Phi_0(\xi) \\ &= e^{-t|\xi|^{2\delta}}\frac{e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}(1-e^{-2t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}})}{2\left|\xi\right|^{2\delta}\sqrt{1-\left|\xi\right|^{2(\sigma-2\delta)}}}\Phi_0(\xi). \end{aligned}$$

We may write $\Phi_0(\xi)=\Phi_0^2(\xi)$. Thus the above symbol can be written as $m_{T0}=m_4m_5m_6$, where

$$m_4=\frac{e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}}{\sqrt{1-\left|\xi\right|^{2(\sigma-2\delta)}}}\Phi_0(\xi),$$

$$m_5=\frac{(1-e^{-2t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}})}{2}\Phi_0(\xi),$$

$$m_6=\frac{1}{|\xi|^{2\delta}}.$$

Note that $m_4=\frac{e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}}{\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}\Phi_0(\xi)$ is exactly the same as the symbol of $S_{s,0}(t,\cdot)*$. So we just need to show one more lemma about the multiplier $m_5(t,\xi)$.

Lemma 5 *For any multi-index α,*

$$\left|\frac{\partial^\alpha}{\partial\xi^\alpha}m_5(t,\xi)\right|\le C\left|\xi\right|^{-|\alpha|},$$

where the constant C is independent of t.

Proof. Again, by taking the multi-index derivative and employing the property of Φ_0, we obtain

$$\left|\frac{\partial^\alpha}{\partial\xi^\alpha}m_5(t,\xi)\right|\le C(t|\xi|^{2\delta-1})^{|\alpha|}e^{-2t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}.$$

Since $e^{-2t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}\le C(t\left|\xi\right|^{2\delta}\sqrt{1-\left|\xi\right|^{2(\sigma-2\delta)}})^{-r}$ for any non-negative real number r, choosing $r=|\alpha|$ yields our desired result. So the proof of Lemma 5 is finished.

By Lemma 5 and Corollary 1, we have the boundedness of $\|T_0(t,\cdot)*u_1(\cdot)\|_{\dot F^{\beta}_{p*,q}}$, such that

$$\begin{aligned}\|T_0(t,\cdot)*u_1(\cdot)\|_{\dot F^{\beta}_{p*,q}} &= \left\|T_{m_{T_0}}u_1\right\|_{\dot F^{\beta}_{p*,q}}\\ &= \|T_{m_4}T_{m_5}T_{m_6}u_1\|_{\dot F^{\beta}_{p*,q}}\\ &\preceq \|T_{m_4}T_{m_6}u_1\|_{\dot F^{\beta}_{p*,q}}\\ &\preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)}t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}}\|T_{m_6}u_1\|_{\dot F^{\gamma}_{p,r}}\\ &\preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)}t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}}\|u_1\|_{\dot F^{\tilde\gamma}_{\tilde p,r}},\end{aligned} \tag{4.28}$$

where $\frac{1}{\tilde p}=\frac{2\delta+\tilde\gamma-\gamma}{n}+\frac{1}{p}$.

Next, we study the boundedness of $\|T_\infty(t,\cdot)*u_1(\cdot)\|_{\dot F^{\beta}_{p*,q}}$.

The symbol of $T_\infty(t,\cdot)*$ is

$$\begin{aligned}m_{T\infty} &= \frac{e^{-t|\xi|^{2\delta}}\sinh\left(t\,|\xi|^{2\delta}\,\mu\right)}{|\xi|^{2\delta}\,\mu}\Phi_\infty(\xi)\\ &= \frac{e^{-t|\xi|^{2\delta}}\sinh\left(t\,|\xi|^{2\delta}\,\mu\right)}{\mu}\Phi_\infty(\xi)\frac{1}{|\xi|^{2\delta}}\\ &= m_6m_7,\end{aligned}$$

where

$$m_7=\frac{e^{-t|\xi|^{2\delta}}\sinh\left(t\,|\xi|^{2\delta}\,\mu\right)}{\mu}\Phi_\infty(\xi),$$

and

$$m_6=\frac{1}{|\xi|^{2\delta}}.$$

Note that m_7 is the same as the symbol of $S_{s,\infty}(t,\cdot)*$, we may immediately derive the boundedness of $\|T_\infty(t,\cdot)*u_1(\cdot)\|_{\dot F^{\beta}_{p*,q}}$, that is

$$\|T_\infty(t,\cdot)*u_1(\cdot)\|_{\dot F^{\beta}_{p*,q}}\preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)}t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}}\|u_1\|_{\dot F^{\tilde\gamma}_{\tilde p,r}}, \tag{4.29}$$

and also we have

$$\|T_1(t,\cdot)*u_1(\cdot)\|_{\dot F^{\beta}_{p*,q}}\preceq t^{-\frac{n}{2(\sigma-\delta)}(1/p-1/p_*)}t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}}\|u_1\|_{\dot F^{\tilde\gamma}_{\tilde p,r}}, \tag{4.30}$$

where $\frac{1}{\tilde p}=\frac{2\delta+\tilde\gamma-\gamma}{n}+\frac{1}{p}$.
Combing (4.42)-(4.30), we finish the proof of (4.12). So the case $p<p_*$, in Proposition 3, is proved. For $p=p_*$ case, we will omit the proof since we can drive the result easily from the

lifting property in Triebel-Liorkin spaces.

Now we start to deal with the case II with $\delta < \sigma < 2\delta$, which is non-effective case. In general, this case is more difficult, since the non diffusive structure appearing at low frequencies is related to the long time decay estimates. For this reason, it is convenient to consider several cases based on the indices of Triebel-Lizorkin spaces $\dot{F}^{\beta}_{p_*,q}$.
We now in a position to sate our second main result.
Case II non-effective case: $\delta < \sigma < 2\delta$.

Proposition 4 *Suppose $\delta < \sigma < 2\delta$, $0 < p \leq p_* \leq q \leq 2$, or $0 \leq p_* \leq q \leq 2$ with $p_* < 1$ or $2 \leq q \leq p_* < \infty$ and $u(t,\cdot)$ is the solution of (4.1). We have*

$$\|u(t,\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n|1/p_*-1/2|(1-\frac{\sigma}{2\delta})} t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} (\|u_0\|_{\dot{F}^{\gamma}_{p,r}} + t^{-(1-\frac{\sigma}{2\delta})}\|u_1\|_{\dot{F}^{\tilde{\gamma}}_{\tilde{p},r}}) \; p_* > p, \tag{4.31}$$

and

$$\|u(t,\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n|1/p_*-1/2|(1-\frac{\sigma}{2\delta})} (\|u_0\|_{\dot{F}^{\gamma}_{p,r}} + t^{-(1-\frac{\sigma}{2\delta})}\|u_1\|_{\dot{F}^{\tilde{\gamma}}_{\tilde{p},r}}) \; p_* = p, \tag{4.32}$$

where $\frac{1}{\tilde{p}} = \frac{2\delta+\tilde{\gamma}-\gamma}{n} + \frac{1}{p}$, and $t > 1$.

Again we shall treat only the case $p_* > p$. To prove inequality (4.31), since $u(t,\cdot) = S(t,\cdot) * u_0(\cdot) + T(t,\cdot) * u_1(\cdot)$, it suffices to prove the following two inequalities respectively

$$\|S(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n|1/p_*-1/2|(1-\frac{\sigma}{2\delta})} t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}}, \tag{4.33}$$

and

$$\|T(t,\cdot) * u_1(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n|1/p_*-1/2|(1-\frac{\sigma}{2\delta})} t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} t^{-(1-\frac{\sigma}{2\delta})} \|u_1\|_{\dot{F}^{\tilde{\gamma}}_{\tilde{p},r}}, \tag{4.34}$$

Since the proofs of (4.33) and (4.34) in Proposition 4 are quite similar. Here we just give the proof of (4.33) which only involves the convolution operator S. Additionally, we just focus on the lower frequency and higher frequency terms.
Proof of (4.33). Notice that

$$\begin{aligned} & e^{-t|\xi|^{2\delta}} \cosh\left(t\,|\xi|^{2\delta}\,\mu\right) \\ = \;& e^{-t|\xi|^{2\delta}} \frac{e^{t|\xi|^{2\delta}\mu} + e^{-t|\xi|^{2\delta}\mu}}{2}. \end{aligned}$$

For simplicity, we may assume the symbol of $S_c(t,\cdot)*$ is

$$e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu}.$$

Now we write

$$e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu} = e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu} \left(\Phi_0(\xi) + \Phi_1(\xi) + \Phi_\infty(\xi)\right).$$

By this decomposition, we have

$$S_c(t,\cdot) * u_0(\cdot) = S_{c,0}(t,\cdot) * u_0(\cdot) + S_{c,1}(t,\cdot) * u_0(\cdot) + S_{c,\infty}(t,\cdot) * u_0(\cdot)$$

where for $j = 0, 1, \infty$ and $S_{c,j}(t,\cdot)*$ has the symbol

$$e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu} \Phi_j(\xi).$$

We now need to prove that

$$\|S_{c,j}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n|1/p_*-1/2|(1-\frac{\sigma}{2\delta})} t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^{\gamma}_{p,r}},$$

for $j = 0, 1, \infty$.

Estimate of $S_{c,0}(t,\cdot) * u_0(\cdot)$. For $|\xi| \leq 1/2$, since $2\delta > \sigma > \delta$, we have

$$\mu(\xi) = i\,|\xi|^{(\sigma-2\delta)} \sqrt{1 - |\xi|^{2(2\delta-\sigma)}}.$$

Therefore, we have that, for $|\xi| \leq 1/2$,

$$\begin{aligned} e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu} &= e^{-t|\xi|^{2\delta}} e^{it|\xi|^{2\delta}|\xi|^{(\sigma-2\delta)}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}} \\ &= e^{-\frac{t}{2}|\xi|^{2\delta}} \left(e^{-\frac{t}{2}|\xi|^{2\delta}} e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}\right). \end{aligned}$$

Thus, we may write

$$\begin{aligned} m_8(t,\xi) &= e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu} \Phi_0(\xi) \\ &= e^{-\frac{t}{2}|\xi|^{2\delta}} e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}} \Phi_0(\xi) e^{-\frac{t}{2}|\xi|^{2\delta}} (t|\xi|^{2\delta})^{\tilde{\beta}_4} \frac{1}{(t|\xi|^{2\delta})^{\tilde{\beta}_4}} \\ &= m_{8,1}(t,\xi) m_{8,2}(t,\xi) m_{8,3}(t,\xi), \end{aligned}$$

where

$$m_{8,1}(t,\xi) = \left(e^{-\frac{t}{2}|\xi|^{2\delta}} e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}\right) \Phi_0(\xi),$$

$$m_{8,2}(t,\xi) = e^{-\frac{t}{2}|\xi|^{2\delta}} (t|\xi|^{2\delta})^{\tilde{\beta}_4},$$

and

$$m_{8,3}(t,\xi) = \frac{1}{(t|\xi|^{2\delta})^{\tilde{\beta}_4}}.$$

Before we continue, we need the following lemma.

Lemma 6 *For any multi-index* α,

$$\left|\frac{\partial^\alpha}{\partial \xi^\alpha} m_{8,1}(t,\xi)\right| \le C t^{|\alpha|(1-\frac{\sigma}{2\delta})} |\xi|^{-|\alpha|},$$

where C *is independent of* t.

Proof. Taking the α-order derivative of $e^{it|\xi|^\sigma\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}$ yields one factor containing the following term

$$(t|\xi|^{\sigma-1})^{|\alpha|} = t^{|\alpha|}|\xi|^{\sigma|\alpha|}|\xi|^{-|\alpha|}.$$

Noting that $e^{-\frac{t}{2}|\xi|^{2\delta}}(\frac{t}{2}|\xi|^{2\delta})^r \le C$ for any non-negative real number r. Let r be $\frac{\sigma|\alpha|}{2\delta}$, we will get the desired result which ends the proof of Lemma 6 .

Next we estimate the boundedness of $\|S_{c,0}(t,\cdot)*u_0(\cdot)\|_{\dot F^\beta_{p_*,q}}$ by considering several cases. Now assume $p_*\in(1,2]$. Since $H^{p_*}=\dot F^0_{p_*,2}$, by Theorem 4 in real Hardy space with $A=t^{1-\frac{\sigma}{2\delta}}$, we have the same conclusion for $\dot F^0_{p_*,2}$, that is

$$\|T_{m_{8,1}}f\|_{\dot F^0_{p_*,2}} \le C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})}\|f\|_{\dot F^0_{p_*,2}}. \tag{4.35}$$

By the definition of Triebel-Lizorkin spaces, we get

$$\begin{aligned}
\|T_{m_{8,1}}u_0\|^{p_*}_{\dot F^0_{p_*,p_*}} &= \left\|\left(\sum_{k\in\mathbb{Z}}|\varphi_k * T_{m_{8,1}}u_0|^{p_*}\right)^{1/p_*}\right\|^{p_*}_{L^{p_*}}\\
&= \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|\varphi_k * T_{m_{8,1}}u_0|^{p_*}dx\\
&= \sum_{k\in\mathbb{Z}}\|T_{m_{8,1}}(\varphi_k * u_0)\|^{p_*}_{L^{p_*}}\\
&\le C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})}\sum_{k\in\mathbb{Z}}\|\varphi_k * u_0\|^{p_*}_{L^{p_*}}\\
&= C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})}\|u_0\|^{p_*}_{\dot F^0_{p_*,p_*}},
\end{aligned}$$

where we used the fact that $H^{p_*}=L^{p_*}$ $(1<p_*<\infty)$. Thus, we obtain that

$$\|T_{m_{8,1}}u_0\|_{\dot F^0_{p_*,p_*}} \le C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})}\|u_0\|_{\dot F^0_{p_*,p_*}}. \tag{4.36}$$

An interpolation yields that

$$\|T_{m_{8,1}}u_0\|_{\dot F^0_{p_*,q}} \le C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})}\|u_0\|_{\dot F^0_{p_*,q}}, \tag{4.37}$$

for any $1<p_*\le q\le 2$.

If $p_* \in (0,1]$, again by Theorem 4 in real Hardy space with $A = t^{1-\frac{\sigma}{2\delta}}$, we have

$$\|T_{m_{8,1}} u_0\|_{\dot{F}^0_{p_*,2}} \leq C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})} \|u_0\|_{\dot{F}^0_{p_*,2}}. \tag{4.38}$$

By the definition of Triebel-Lizorkin spaces, we have

$$\begin{aligned}
\|T_{m_{8,1}} u_0\|^{p_*}_{\dot{F}^0_{p_*,p_*}} &= \left\| \Big(\sum_{k\in\mathbb{Z}} |\varphi_k * T_{m_{8,1}} u_0|^{p_*}\Big)^{1/p_*} \right\|^{p_*}_{L^{p_*}} \\
&= \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\varphi_k * T_{m_{8,1}} u_0|^{p_*} dx \\
&= \sum_{k\in\mathbb{Z}} \|T_{m_{8,1}}(\varphi_k * u_0)\|^{p_*}_{L^{p_*}} \\
&\leq \sum_{k\in\mathbb{Z}} \|T_{m_{8,1}}(\varphi_k * u_0)\|^{p_*}_{H^{p_*}} \\
&\leq C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})} \sum_{k\in\mathbb{Z}} \|\varphi_k * u_0\|^{p_*}_{H^{p_*}}.
\end{aligned}$$

We now pass to the Triebel-Lizorkin norm by using the Riesz transform characterization (see Appendix),

$$\sum_{k\in\mathbb{Z}} \|\varphi_k * u_0\|^{p_*}_{H^{p_*}} \leq \sum_J \sum_{k\in\mathbb{Z}} \|R_J \varphi_k * u_0\|^{p_*}_{L^{p_*}}, \tag{4.39}$$

where $\sum_J$ is a sum of finite terms. Now since the definition of Triebel-Lizorkin space is independent of the choice of $\{\varphi_k\}$, it is easy to check that $\{\tilde{\varphi}_k\} = \{R_J \varphi_k\} = \{(R_J \varphi)_k\}$ is another family of generating functions. Thus we obtain that

$$\|T_{m_{8,1}} u_0\|_{\dot{F}^0_{p_*,p_*}} \leq C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})} \|u_0\|_{\dot{F}^0_{p_*,p_*}}. \tag{4.40}$$

An interpolation yields that

$$\|T_{m_{8,1}} u_0\|_{\dot{F}^0_{p_*,q}} \leq C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*}-\frac{1}{2})} \|u_0\|_{\dot{F}^0_{p_*,q}}. \tag{4.41}$$

for any $0 < p_* \leq q \leq 2$ with $p_* \leq 1$.

Using duality, for any $2 \le q \le p_* < \infty$, we have that

$$\begin{aligned}
\|T_{m_{8,1}} u_0\|_{\dot{F}^0_{p_*,q}} &= \sup_{\|g\|_{\dot{F}^0_{p_*^{'},q^{'}}} \le 1} | < T_{m_{8,1}} u_0, g > | \\
&= \sup_{\|g\|_{\dot{F}^0_{p_*^{'},q^{'}}} \le 1} | < u_0, T_{m_{8,1}} g > | \\
&\preceq \sup_{\|g\|_{\dot{F}^0_{p_*^{'},q^{'}}} \le 1} \|u_0\|_{\dot{F}^0_{p_*,q}} \|T_{m_{8,1}} g\|_{\dot{F}^0_{p_*^{'},q^{'}}} \\
&\le C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{p_*^{'}}-\frac{1}{2})} \|u_0\|_{\dot{F}^0_{p_*,q}} \\
&= C t^{n(1-\frac{\sigma}{2\delta})(\frac{1}{2}-\frac{1}{p_*})} \|u_0\|_{\dot{F}^0_{p_*,q}}.
\end{aligned}$$

As usual, $p_*^{'}$ and $q^{'}$ are the conjugates of p_* and q separately. We applied the duality property, in the first inequality, in the homogenous Triebel-Lizorkin spaces (see Appendix).

Thus, for $1 < p_* \le q \le 2$, or $0 < p_* \le q \le 2$ with $p_* \le 1$, we have

$$\begin{aligned}
\|S_{c,0}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}} &= \|T_{m_8} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&= \|T_{m_{8,1}} T_{m_{8,2}} T_{m_{8,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&\preceq t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})} \|T_{m_{8,2}} T_{m_{8,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&\preceq t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})} \|T_{m_{8,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&\preceq t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})} t^{-\tilde{\beta}_4} \|u_0\|_{\dot{F}^\gamma_{p,r}} \\
&= t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})} t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^\gamma_{p,r}},
\end{aligned}$$

where $\tilde{\beta}_4$ is determined by $\beta - n/p_* = 2\delta\tilde{\beta}_4 + \gamma - n/p$.

Also, we have, as $2 \le q \le p_* < \infty$,

$$\begin{aligned}
\|S_{c,0}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}} &= \|T_{m_8} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&= \|T_{m_{8,1}} T_{m_{8,2}} T_{m_{8,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&\preceq t^{n(1/2-1/p_*)(1-\frac{\sigma}{2\delta})} \|T_{m_{8,2}} T_{m_{8,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&\preceq t^{n(1/2-1/p_*)(1-\frac{\sigma}{2\delta})} \|T_{m_{8,3}} u_0\|_{\dot{F}^\beta_{p_*,q}} \\
&\preceq t^{n(1/2-1/p_*)(1-\frac{\sigma}{2\delta})} t^{-\tilde{\beta}_4} \|u_0\|_{\dot{F}^\gamma_{p,r}} \\
&= t^{n(1/2-1/p_*)(1-\frac{\sigma}{2\delta})} t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})} t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}} \|u_0\|_{\dot{F}^\gamma_{p,r}}.
\end{aligned}$$

Estimate of $\|S_{c,\infty}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^\beta_{p_*,q}}$. Note that the symbol of $S_{c,\infty}(t,\cdot)*$ is $e^{-t|\xi|^{2\delta}} e^{t|\xi|^{2\delta}\mu}\Phi_\infty(\xi)$. When $|\xi| \ge 2$, $\delta < \sigma < 2\delta$, then $\mu(\xi) = \sqrt{1-|\xi|^{2(\sigma-2\delta)}}$. So this estimate should be the same as $e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu}\Phi_0(\xi)$ in the case $\sigma-2\delta>0$.

Finally, to complete the proof of Proposition 4, we have to show

$$\|S_s(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n|1/p_*-1/2|(1-\frac{\sigma}{2\delta})}t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})}t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}}\|u_0\|_{\dot{F}^{\gamma}_{p,r}}.$$

Indeed, we may get a better decay, namely,

$$\|S_s(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n|1/p_*-1/2|(1-\frac{\sigma}{2\delta})}t^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{p}-\frac{1}{p_*})}t^{-\frac{\beta-\gamma}{2(\sigma-\delta)}}t^{-(1-\frac{\sigma}{2\delta})}\|u_0\|_{\dot{F}^{\gamma}_{p,r}}.$$

Estimate of $\|S_{s,0}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}}$. For simplicity, as we did with $S_{c,0}(t,\cdot)*$, we may assume the symbol of $S_{s,0}(t,\cdot)*$ is

$$\begin{aligned}
m_9(t,\xi) &= \frac{e^{-t|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu}}{\mu}\Phi_0(\xi)\\
&= \frac{e^{-t|\xi|^{2\delta}}e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}}{i\,|\xi|^{(\sigma-2\delta)}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}\Phi_0(\xi)\\
&= e^{-\frac{t}{2}|\xi|^{2\delta}}(t|\xi|^{2\delta})^{\tilde{\beta_5}}\frac{1}{(t|\xi|^{2\delta})^{\tilde{\beta_5}}}\frac{e^{-\frac{t}{2}|\xi|^{2\delta}}e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}}{i\,|\xi|^{(\sigma-2\delta)}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}\Phi_0(\xi)\\
&= m_{9,1}(t,\xi)m_{9,2}(t,\xi)m_{9,3}(t,\xi),
\end{aligned}$$

where

$$\begin{aligned}
m_{9,1}(t,\xi) &= \left(\frac{e^{-\frac{t}{2}|\xi|^{2\delta}}e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}}{i\,|\xi|^{(\sigma-2\delta)}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}\right)\Phi_0(\xi)\\
&= \frac{m_{8,1}(t,\xi)}{i\,|\xi|^{(\sigma-2\delta)}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}},
\end{aligned}$$

$$m_{9,2}(t,\xi)=e^{-\frac{t}{2}|\xi|^{2\delta}}(t|\xi|^{2\delta})^{\tilde{\beta_5}},$$

and

$$m_{9,3}(t,\xi)=\frac{1}{(t|\xi|^{2\delta})^{\tilde{\beta_5}}}.$$

As we did in Lemma 6 , after we take the α-order derivative of $e^{it|\xi|^{\sigma}\sqrt{1-|\xi|^{2(2\delta-\sigma)}}}$ contained in $m_{9,1}$, there is a factor containing $(t|\xi|^{\sigma-1})^{|\alpha|}=t^{|\alpha|}|\xi|^{\sigma|\alpha|}|\xi|^{-|\alpha|}$. Since $|e^{-\frac{t}{2}|\xi|^{2\delta}}(\frac{t}{2}|\xi|^{2\delta})^r|\le C$ for any non-negative real number r, in order to make $m_{9,1}$ satisfy the conditions of multiplier theorem, we choose r to be $\frac{\sigma|\alpha|+2\delta-\sigma}{2\delta}$. Then, we obtain that

$$\left|\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}m_{9,1}(t,\xi)\right|\le t^{|\alpha|(1-\frac{\sigma}{2\delta})}t^{-(1-\frac{\sigma}{2\delta})}\,|\xi|^{-|\alpha|}.$$

Applying Theorem 4 and the product rule of derivatives in higher dimension, following the similar steps to show the boundedness of $\|S_{c,0}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p*,q}}$, we get the desired estimates of $\|S_{s,0}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p*,q}}$ such that

$$\|S_{s,0}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p*,q}} \preceq t^{n(1/q-1/2)(1-\frac{\sigma}{2\delta})}t^{-\frac{1}{2\delta}(n(1/p-1/q)}t^{-(1-\frac{\sigma}{2\delta})}\|u_0\|_{\dot{F}^{\gamma}_{p,r}}.$$

Thus, for $1<p_*\le q\le 2$, or $0<p_*\le q\le 2$ with $p_*\le 1$, we have

$$\begin{aligned}
\|S_{s,0}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p*,q}} &= \|T_{m_9}u_0\|_{\dot{F}^{\beta}_{p*,q}} = \|T_{m_{9,1}}T_{m_{9,2}}T_{m_{9,3}}u_0\|_{\dot{F}^{\beta}_{p*,q}}\\
&\preceq t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})}t^{-(1-\frac{\sigma}{2\delta})}\|T_{m_{9,2}}T_{m_{9,3}}u_0\|_{\dot{F}^{\beta}_{p*,q}}\\
&\preceq t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})}t^{-(1-\frac{\sigma}{2\delta})}\|T_{m_{9,3}}u_0\|_{\dot{F}^{\beta}_{p*,q}}\\
&\preceq t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})}t^{-(1-\frac{\sigma}{2\delta})}t^{-\tilde{\beta_5}}\|u_0\|_{\dot{F}^{\gamma}_{p,r}}\\
&= t^{n(1/p_*-1/2)(1-\frac{\sigma}{2\delta})}t^{-(1-\frac{\sigma}{2\delta})}t^{-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{p_*})}t^{-\frac{\beta-\gamma}{2\delta}}\|u_0\|_{\dot{F}^{\gamma}_{p,r}},
\end{aligned}$$

where $\tilde{\beta_5}$ is determined by $\beta-n/p_*=2\delta\tilde{\beta_5}+\gamma-n/p$.

Estimate of $\|S_{s,\infty}(t,\cdot)*u_0(\cdot)\|_{\dot{F}^{\beta}_{p*,q}}$. Note that the symbol of the operator $S_{s,\infty}(t,\cdot)*$ is

$$\begin{aligned}
&\frac{e^{-t|\xi|^{2\delta}}sinh(t|\xi|^{2\delta}\mu)}{\mu}\Phi_\infty(\xi)\\
&= \frac{e^{-t|\xi|^{2\delta}}(e^{t|\xi|^{2\delta}\mu}-e^{-t|\xi|^{2\delta}\mu})}{2\mu}\Phi_\infty(\xi)\\
&= e^{-\frac{t}{2}|\xi|^{2\delta}}\frac{e^{-\frac{t}{2}|\xi|^{2\delta}}(e^{t|\xi|^{2\delta}\mu}-e^{-t|\xi|^{2\delta}\mu})}{2\mu}\Phi_\infty(\xi)\\
&= e^{-\frac{t}{2}|\xi|^{2\delta}}\frac{e^{-\frac{t}{2}|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\mu}}{2\mu}\Phi_\infty(\xi)(1-e^{-2t|\xi|^{2\delta}\mu})\\
&= e^{-\frac{t}{2}|\xi|^{2\delta}}\frac{e^{-\frac{t}{2}|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}}{2\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}\Phi_\infty(\xi)(1-e^{-2t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}})\\
&= m_{10}(t,\xi)m_{11}(t,\xi)m_{12}(t,\xi),
\end{aligned}$$

where

$$m_{10}(t,\xi)=e^{-\frac{t}{2}|\xi|^{2\delta}},$$

$$m_{11}(t,\xi)=\frac{e^{-\frac{t}{2}|\xi|^{2\delta}}e^{t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}}{2\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}\Phi_\infty(\xi),$$

and

$$m_{12}(t,\xi)=1-e^{-2t|\xi|^{2\delta}\sqrt{1-|\xi|^{2(\sigma-2\delta)}}}.$$

As $|\xi| \geq 2$, $\delta < \sigma < 2\delta$, we have

$$\mu = \sqrt{1 - |\xi|^{2(\sigma-2\delta)}} \approx |\xi|^{\sigma-2\delta}.$$

In fact, by the Taylor expansion

$$\sqrt{1 - |\xi|^{2(\sigma-2\delta)}} = 1 - \frac{1}{2}|\xi|^{2(\sigma-2\delta)} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}|\xi|^{2(\sigma-2\delta)} + \cdots$$

So $1 - \mu \approx |\xi|^{2(\sigma-2\delta)}$, which implies $(1-\mu)^{\frac{1}{2}} \approx |\xi|^{(\sigma-2\delta)}$.
Then $\mu = \sqrt{1 - |\xi|^{2(\sigma-2\delta)}} \approx |\xi|^{\sigma-2\delta}$ because of $\mu \approx (1-\mu)^{\frac{1}{2}}$. Technically, we may deal with $m_{11}(t,\xi)$ the same as we did with the symbol of $S_{s,0}(t,\cdot)\,*$.

Thus, we obtain

$$\|S_{s,\infty}(t,\cdot) * u_0(\cdot)\|_{\dot{F}^{\beta}_{p_*,q}} \preceq t^{n(1/q-1/2)_+ + (1-\frac{\sigma}{2\delta})} t^{-\frac{1}{2\delta}(n(1/p-1/q)} t^{-(1-\frac{\sigma}{2\delta})} \|u_0\|_{\dot{F}^{\gamma}_{p,r}},$$

which ends the proof of Proposition 4 for $p_* > p$ in Triebel-Lizorkin spaces.

Remark. For the case $p_* = p$ in Triebel-Lizorkin spaces, we can get the estimates immediately from the known results of multipliers on Triebel-Lizorkin spaces. Following the similar way, we also can get the counterpart for Besov spaces , we omit the details.

Chapter 5

Future Research

(i) Fourier Multipliers with Parameters.

Since the milestone works of S.G. Michilin and L. Hörmander, Fourier multipliers for function spaces have attracted much attention for their own sake. However, due to the restrictions on indices ($1 < p < \infty$) and function spaces ($L^p(\mathbb{R}^n)$), Fourier multipliers remain very much open to investigation. In [25], Miyachi derived optimal estimates of some Fourier multipliers with parameters for real Hardy spaces $H^p(\mathbb{R}^n)$ with ($0 < p < 2$). In 2016, with an application of the results in [25], M.D'Abbico et al obtained the long time decay estimate for the evolution equations with structural dissipation in real Hardy spaces $H^p(\mathbb{R}^n)$. Since $\dot{F}^0_{p,2}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$ with ($0 < p < \infty$), it seems natural to study the Fourier multiplier with parameters in $\dot{F}^s_{p,q}(\mathbb{R}^n)$ spaces which will probably to get the long time decay estimate for the generalized wave equation in Triebel-Lizorkin spaces for all indices.

(ii) Characterizations and Decompositions of Function Spaces

Function spaces have been studied for a long time and are an important part of harmonic analysis and PDEs. Usually different characterizations or decompositions of function spaces provide different advantages. For example, in [25], the Riesz transform characterization of $H^p(\mathbb{R}^n)$ simplified the core proof to get the optimal estimates of Fourier multiplier with parameters, which is useful to solve our time decay problems. And in [6], $(p, 2)$- atom decomposition of $f \in H^p(\mathbb{R}^n)$ with $0 < p \leq 1$ guarantees the upper boundedness of the $|\hat{f}(\xi)|$ (the absolute value of the Fourier transform), which is helpful for obtaining the boundedness of Fourier multiplier operators in real Hardy spaces $H^p(\mathbb{R}^n)$. To the best of my knowledge, so far, there seems no corresponding Riesz transform characterization of Triebel-Lizorkin spaces

$\dot{F}^s_{p,q}(\mathbb{R}^n)$. The Riesz transform characterization of the Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and its applications to derive the optimal estimates of the corresponding Fourier multiplier with parameters will open a new chapter in harmonic analysis and partial differential equations. Furthermore, a very modern characterization of spaces, wavelet characterization, which extends the classical atom decomposition, is also used in studying PDEs [21]. So it is challenging and worthy to study characterizations and decompositions of function spaces.

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Appendix

Firstly, let us recall the characterization of $H^p(\mathbb{R}^n)$ by the Riesz transforms. Let $R_J, J = (j_1, \cdots, j_s) \in \{0, 1, \cdots, n\}^s$, will be the Riesz transform of order s, i.e. the Fourier multiplier transformation T_m with

$$m(\xi) = m_J(\xi) = (-i\frac{\xi_{j_1}}{|\xi|}) \cdots (-i\frac{\xi_{j_s}}{|\xi|}) \quad \xi \in \mathbb{R}^n, \tag{5.1}$$

where the factor $(-i\xi_j/|\xi|)$ should be replaced by 1 if $j = 0$. With this assumption, we have the following theorem.

Theorem 14 *([25]) Let $p > (n-1)/(n-1+s)$. Then $f \in L^2(\mathbb{R}^n) \bigcap H^p(\mathbb{R}^n)$ if and only if $R_J f \in L^2(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$ for all $J \in \{0, 1, \cdots, n\}^s$; and there exist constants C and C' depending only on $p, n,$ and s such that*

$$C \sum_J \|R_J f\|_{L^p} \leq \|f\|_{H^p} \leq C' \|R_J f\|_{L^p}, \qquad f \in L^2(\mathbb{R}^n) \bigcap H^p(\mathbb{R}^n). \tag{5.2}$$

Next we recall an important version of $L^p - L^q$ estimates for fractional integration in the context of Hardy spaces [29]. Let I_r be the Riesz potential with order $r > 0$, defined by means of $I_r f(x) := \mathcal{F}^-(|\xi|^{-r} \mathcal{F} f(\xi))$. We notice that $I_r(I_s f) = I_{r+s} f$. If $r \in (0, n)$, the Riesz potential may be represented for sufficiently smooth f by

$$I_r f(x) = c_{n,r} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-r}} dy, \tag{5.3}$$

for suitable $c_{n,r}$.

Theorem 15 *([6]) Consider $r > 0$ and $0 < p < n/r$. Then, there exists $C = C(r, p) > 0$ such that*

$$\|I_r f\|_{H^q(\mathbb{R}^n)} \leq C \|f\|_{H^p(\mathbb{R}^n)}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{r}{n}.$$

We end this appendix by proving the following duality property in the homogeneous Triebel-Lizorkin spaces which is stated as below.

Proposition 5 *Assume $u \in \dot{F}^s_{p,q}, v \in \dot{F}^{-s}_{p',q'}$, then*

$$|< u, v >| \preceq \|u\|_{\dot{F}^s_{p,q}} \|v\|_{\dot{F}^{-s}_{p',q'}}.$$

To prove the duality property in Triebel-Lizorkin spaces, we need to prove an elementary inequality which is stated as follows:

$$\sum_{|j-j'|\leq 1} |a_j b_{j'}| \leq \|\{a_j\}\|_{l_j^q} \|\{b_j\}\|_{l_{j'}^{q'}} \|1_{|l|\leq 1}\|_{l_l^1} \preceq \|\{a_j\}\|_{l_j^q} \|\{b_j\}\|_{l_{j'}^{q'}} \tag{5.4}$$

$$1_{|l|\leq 1}(k) = \begin{cases} 1, & |k| \leq l, \\ 0, & others, \end{cases} \tag{5.5}$$

where $j, j', k, l \in \mathbb{Z}, l \geq 1$ and $\|\{a_j\}\|_{l^q} := (\sum_{j\in\mathbb{Z}} |a_j|^q)^{\frac{1}{q}}$.

Proof. We have, thanks to Hölder's inequality and Young's inequality in discrete form,

$$\begin{aligned}
\sum_{|j-j'|\leq 1} |a_j b_{j'}| &= \sum_j \Big|a_j \sum_{|j-j'|\leq 1} b_{j'}\Big| \\
&= \sum_j \Big|a_j \sum_{j'} \{b_{j'}\} * \{1_l\}(j)\Big| \qquad (\text{where } l = j - j') \\
&\leq \|\{a_j\}\|_{l_j^q} \|\{b_{j'}\} * \{1_l\}(j)\|_{l_j^{q'}} \qquad (\textit{ Hölder's inequality in discrete form}) \\
&\leq \|\{a_j\}\|_{l_j^q} \|\{b_j\}\|_{l_{j'}^{q'}} \|1_l\|_{l_l^1} \qquad (\text{Young's inquality in discrete form}) \\
&= 3\|\{a_j\}\|_{l_j^q} \|\{b_j\}\|_{l_{j'}^{q'}}.
\end{aligned}$$

To proceed, we also need the following two lemmas.

Lemma 7 *Suppose $u_n \to u$ in L^p, $v_n \to v$ in $L^{p'}$ and $\int_{\mathbb{R}^n} u_n v_m dx = 0$, then we have*

$$\int_{\mathbb{R}^n} uv dx = 0,$$

where $u_n, v_m \in C_c^\infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By the assumption, we have $\|u_n - u\|_{L^p} \to 0$ as $n \to \infty$ and $\|v_m - v\|_{L^{p'}} \to 0$ as $m \to \infty$. Then

$$\lim_{n\to\infty} < u_n, v_m > = < u, v_m >, \qquad (\text{for fixed m}).$$

In fact,

$$| < u_n - u, v_m > | \leq \|u_n - u\|_{L^p} \|v_m\|_{L^{p'}} \to 0, \text{as } n \to \infty.$$

Similarly,

$$\lim_{m\to\infty} \lim_{n\to\infty} < u_n, v_m > = \lim_{m\to\infty} < u, v_m > = < u, v > .$$

So we derive

$$\int_{\mathbb{R}^n} uv dx = < u, v > = \lim_{m\to\infty} \lim_{n\to\infty} < u_n, v_m > = 0.$$

Lemma 8 *Suppose $u_n \to u$ in L^p, then we have*

$$\dot{\Delta}_j u_n \to \dot{\Delta}_j u \quad in\ L^p,$$

where $u_n \in C_c^\infty$, and $\dot{\Delta}_j u := \varphi_j * u$.

Proof. By the definition of $\dot{\Delta}_j u := \varphi_j * u$ with $\varphi_j(x) = 2^{jn} h(2^j x)$, we have

$$\begin{aligned} \|\dot{\Delta}_j u_n - \dot{\Delta}_j u\|_{L^p} &= \|\varphi_j * u_n - \varphi_j * u\|_{L^p} \\ &= \|\varphi_j * (u_n - u)\|_{L^p} \\ &\leq \|\varphi_j\|_{L^1} \|u_n - u\|_{L^p} \to 0, \text{ as } n \to \infty, \end{aligned}$$

where, in the last step, we used Young's inequality and the fact that $\varphi_j \in L^1$.

Now we are in the position to prove the duality Proposition 5 in Triebel-Lizorkin spaces.

Proof. By the assumption, we have $2^{-js}\dot{\Delta}_j u \in L^p$, $2^{j's}\dot{\Delta}'_j v \in L^{p'}$. Thus, for $|j - j'| \leq 1$, we have

$$\begin{aligned} |< u, v >| &= |< \sum_j \dot{\Delta}_j u, \sum_{j'} \dot{\Delta}_{j'} v >| \\ &= \int_{\mathbb{R}^n} \sum_{|j-j'|\leq 1} 2^{(-j+j')s} 2^{js} \dot{\Delta}_j u 2^{-j's} \dot{\Delta}_{j'} v dx \\ &\leq \int_{\mathbb{R}^n} \sum_{|j-j'|\leq 1} 2^{|-j+j'||s|} |2^{js} \dot{\Delta}_j u| |2^{-j's} \dot{\Delta}_{j'} v| dx \\ &\leq 2^{|s|} \int_{\mathbb{R}^n} \sum_{|j-j'|\leq 1} |2^{js} \dot{\Delta}_j u| |2^{-j's} \dot{\Delta}_{j'} v| dx \quad (\text{since } |j - j'| \leq 1) \\ &= 2^{|s|} \int_{\mathbb{R}^n} \sum_{|j-j'|\leq 1|} |a_j b_{j'}| dx \quad (\text{where } a_j = \{2^{js} \dot{\Delta}_j u\} b_{j'} = \{2^{-j's} \dot{\Delta}_{j'} v\}) \\ &\leq 2^{|s|} \int_{\mathbb{R}^n} \|\{a_j\}\|_{l^q_j} \|\{b_j\}\|_{l^{q'}_{j'}} \|1_{|l|\leq 1}\|_{l^1_l} dx \quad (\text{by the elementary inequality above}) \\ &\preceq 3 \cdot 2^{|s|} \int_{\mathbb{R}^n} \|\{a_j\}\|_{l^q_j} \|\{b_j\}\|_{l^{q'}_{j'}} dx \\ &\leq 3 \cdot 2^{|s|} \Big\| \|\{a_j\}\|_{l^q_j} \Big\|_{L^p} \Big\| \|\{b_j\}\|_{l^{q'}_{j'}} \Big\|_{L^{p'}} \quad (\text{Hölder's inequality}) \\ &= 3 \cdot 2^{|s|} \|u\|_{\dot{F}^s_{p,q}} \|v\|_{\dot{F}^{-s}_{p',q'}}. \end{aligned}$$

Remark. The same result holds for Besov spaces. We refer the reader to [1] for the details.

Curriculum Vitae

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Place of birth: Henan, China

EDUCATION

University of Wisconsin-Milwaukee

Ph.D. in Mathematics, December 2018

- Thesis Title: Triebel-Lizorkin spaces estimates for evolution equations with structure dissipation
- Advisor: Professors Dashan Fan and Lijing Sun

Central China Normal University, Wuhan, China

M.S. in Mathematics, June 2013

- Thesis Title: The existence of nontrivial solutions for a class of semilinear Schrődinger equations with vanishing potentials
- Advisor: Professor Yinbin Deng

Zhoukou Normal School, Henan, China

B.S. in Mathematics, June 2010

RESEARCH INTERESTS

Harmonic analysis and partial differential equations, especially about how to use harmonic tools to study partial differential equations.

PULICATIONS & PREPRINTS

- Triebel-Lizorkin space estimates for evolution equations with structure dissipation, with Dashan Fan, submitted to J. Pseudo-Differ. Oper. Appl.

- Besov compactness of time-dependent kinetic equation, with Cong He, submitted to Mathematische Zeitschrift.
- L2 solution of linear non-cutoff Boltzmann equation with boundary conditions, with Cong He, submitted to JMAA.
- On local existence of the Vlasov-Fokker-Planck equation in a 2D anisotropic space, with Cong He, published to BVP. http://wwwboundaryvalueproblems.com/content/2013/1/233.

CONFERENCES ATTENDED

- Joint Mathematics Meetings in San Diego, CA (presenter) January 2018
- Teaching and Learning Symposium, University of Wisconsin-Milwaukee January 2017
- Annual MAA Wisconsin section meeting, University of Wisconsin-Milwaukee April 2017
- Partial Differential Equations and Variations Method meeting, Central China Normal University May 2014

TEACHING

University of Wisconsin-Milwaukee

MATH 231 Calculus & Analytic Geometry I	Instructor	Spring 2018
MATH 231 Calculus & Analytic Geometry I	Instructor	Fall 2017
MATH 116 College Algebra	Instructor	Spring 2017
MATH 211 Survey-Calc/Analytic Geometry	Discussion	Fall 2016
MATH 105 Intermediate Algebra (Online)	Instructor	Summer 2016
MATH 105 Intermediate Algebra (Online)	Instructor	Spring 2016

- MATH 105 Intermediate Algebra Instructor Fall 2015
- MATH 233 Calculus & Analytic Geometry III Grader Summer 2015
- MATH 102 Math Litrcy: College Student II TA Spring 2015
- MATH 092 Math Litrcy: College Student I TA Fall 2014

Central China Normal University, Wuhan, China

- MATH 231 Calculus & Analytic Geometry II TA Spring 2012

Off Campus, China

- Private tutor in math for middle/senior high school student 2010-2014
- Teaching English for middle high school student Summer 2011 & 2012
- Lecture in Math and English in an Elementary School Spring 2010

AWARDS

- Research Assistantship

  University of Wisconsin-Milwaukee Summer 2018
- Ernst Schwandt Memorial Scholarship and Ernst Schwandt Graduate Teaching Award in recognition of exceptional teaching achievement

  University of Wisconsin-Milwaukee Spring 2018
- Graduate Student Travel Grant to the Joint Mathematics Meetings

  American Mathematical Society January 2018
- Department of Mathematics Sciences Student Conference Travel Award

  University of Wisconsin-Milwaukee Fall 2017
- Graduate Student Travel Award (Graduate School)

  University of Wisconsin-Milwaukee Fall 2017
- Mark Lawrence Teply Award in recognition of outstanding research potential (two times)

  University of Wisconsin-Milwaukee Spring 2017 & 2018
- Chancellor’s Graduate Student Award

University of Wisconsin-Milwaukee 2014-Present

- Excellent Student Scholarship (3 times)

Central China Normal University 2010-2013

SKILLS & CERTIFICATES

- Knowledge in Language C (Grade:2); Network Technique (Grade: 3); Latex
- Proficient in Microsoft Word, Excel, PowerPoint
- Proficient in oral/written English and native in Chinese